

Your Name: *Answer Key*

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Draw a box around your final answer. You must show all work to receive credit.

1. Power Series (8.7)

(a: #4) Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$.

Abs Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \frac{(3x-2)^{n+1}}{(3x-2)^n} \right| = |3x-2| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = |3x-2|$$

Int of Convergence

Converges for $-1 < 3x-2 < 1 \Rightarrow 1 < 3x < 3 \Rightarrow \boxed{\frac{1}{3} \leq x < 1}$

$x = \frac{1}{3}$

$$\sum \frac{(1-2)^n}{n} = \sum (-1)^n \frac{1}{n}$$

Converges!

$x = 1$

$$\sum \frac{(3-2)^n}{n} = \sum \frac{1^n}{n}$$

diverges

Radius of convergence: $\frac{1 - \frac{1}{3}}{2} = \frac{\frac{2}{3}}{2}$

$\boxed{\frac{1}{3}}$

(b: #14) Find the interval and radius of convergence for $\sum_{n=0}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$.

Abs Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(2x+3)^{2(n+1)+1}}{(n+1)!}}{\frac{(2x+3)^{2n+1}}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x+3)^{2n+3}}{(2x+3)^{2n+1}} \cdot \frac{n!}{(n+1)!} \right| = (2x+3)^2 \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0$$

Converges for all x OR $\boxed{-\infty < x < \infty}$

Radius $\boxed{\infty}$

(c: Based on #44)

The exponential function e^x may be represented by the power series (with interval of convergence $-\infty < x < \infty$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots$$

Using term-by-term differentiation and integration, show that e^x has a derivative of e^x and an integral of $e^x + C$.

$$\begin{aligned} \frac{d}{dx}(e^x) &= \sum_{n=0}^{\infty} \frac{d}{dx} \left(\frac{x^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = 0 + \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \end{aligned}$$

$$\begin{aligned} \int e^x dx &= \sum_{n=0}^{\infty} \int \frac{x^n}{n!} dx + C \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{n+1} x^{n+1} + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} + C \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n!} + C = \sum_{n=1}^{\infty} \frac{x^n}{n!} + \frac{x^0}{0!} + C = \sum_{n=0}^{\infty} \frac{x^n}{n!} + C = e^x + C \end{aligned}$$

(See 8.7 #1-38 and #41 for more examples of questions about Power Series.)

2. Taylor and Maclaurin Series (8.8)

(a: #2) Find the Taylor Polynomial $P_3(x)$ of order 3 generated by $f(x) = \frac{1}{1+x}$ at $x = a$.

$$f^{(0)}(x) = (1+x)^{-1}$$

$$f^{(1)}(x) = -(1+x)^{-2}$$

$$f^{(2)}(x) = 2(1+x)^{-3}$$

$$f^{(3)}(x) = -6(1+x)^{-4}$$

$$P_3(x) = (1+a)^{-1} - (1+a)^{-2}x + \frac{2(1+a)^{-3}}{2!}x^2 - \frac{6(1+a)^{-4}}{3!}x^3$$

$$= \boxed{(1+a)^{-1} - (1+a)^{-2}x + (1+a)^{-3}x^2 - (1+a)^{-4}x^3}$$

(b: #14) Find the Maclaurin Series generated by $\sin\left(\frac{x}{2}\right)$.

$$f^{(0)}(x) = \sin\left(\frac{x}{2}\right)$$

$$f^{(1)}(x) = \cos\left(\frac{x}{2}\right) \cdot \left(\frac{1}{2}\right)$$

$$f^{(2)}(x) = -\sin\left(\frac{x}{2}\right) \cdot \left(\frac{1}{4}\right)$$

$$f^{(3)}(x) = -\cos\left(\frac{x}{2}\right) \cdot \left(\frac{1}{8}\right)$$

$$f^{(4)}(x) = \sin\left(\frac{x}{2}\right) \cdot \left(\frac{1}{16}\right)$$

⋮

$$f^{(2k)}(x) = (-1)^k \left(\frac{1}{2^{2k}}\right) \sin\left(\frac{x}{2}\right)$$

$$f^{(2k+1)}(x) = (-1)^k \left(\frac{1}{2^{2k+1}}\right) \cos\left(\frac{x}{2}\right)$$

$$f^{(2k)}(0) = \frac{(-1)^k \left(\frac{1}{2^{2k}}\right) \sin\left(\frac{0}{2}\right)}{2^{2k}} = 0$$

$$f^{(2k+1)}(0) = (-1)^k \left(\frac{1}{2^{2k+1}}\right) \cos\left(\frac{0}{2}\right)$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2^{2k+1}}\right)}{(2k+1)!} x^{2k+1} = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2^{2k+1} (2k+1)!}}$$

(c:) Using the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ with interval of convergence $-1 < x < 1$, find a power series to represent $f(x) = \frac{2}{3(1-x)^2}$, give its interval of convergence, and write out the first four terms of the series.

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$$

$$-(1-x)^{-2} = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{2}{3}(1-x)^{-2} = \frac{2}{3} \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{2}{3(1-x)^2} = \sum_{n=0}^{\infty} \frac{2}{3} n x^{n-1} = 0 + \frac{2}{3}(1)x^0 + \frac{2}{3}(2)x^1 + \frac{2}{3}(3)x^2 + \frac{2}{3}(4)x^3 + \dots$$

$$= \frac{2}{3} + \frac{4}{3}x + 2x^2 + \frac{8}{3}x^3 + \dots$$

Int of Convergence: $(-1 < x < 1)$

(d:) Now find the Taylor Polynomial $P_4(x)$ of order 4 generated by $f(x) = \frac{2}{3(1-x)^2}$ at $x = 0$.

$$P_4(x) = \frac{2}{3} + \frac{4}{3}x + 2x^2 + \frac{8}{3}x^3$$

3. Convergence of Taylor Series (8.9)

(a.) Show that $\cos(x)$ converges to its Taylor Series $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ for any real number x .

Show $\lim_{n \rightarrow \infty} R_n(x) = 0$ $\sin(c_n) \leq 1$ or $\cos(c_n) \leq 1$

$$\lim_{n \rightarrow \infty} \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} x^{n+1} \right| \leq \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)!} x^{n+1} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \right| = 0$$

(b.) Express $\cos(x^2)$ as a power series.

$$\cos(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k}}{(2k)!} = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!}}$$

(c:) Express $\int \cos(x^2) dx$ as a power series.

$$\int \cos(x^2) dx = \sum_{k=0}^{\infty} (-1)^k \frac{1}{4k+1} \frac{x^{4k+1}}{(2k)!} + C$$

(d:) Evaluate $e^{1+3\pi i}$.

$$e^{1+3\pi i} = e^1 (e^{3\pi i}) = e (\cos 3\pi + i \sin 3\pi) = e(-1) = \boxed{-e}$$

(See Section 8.9 #1-18 for more examples of Convergence of Taylor Series examples. Practice finding $\lim_{n \rightarrow \infty} R_n$ in particular.)

4. Binomial Series (8.10)

(a.) Find the first four terms of the expansion of $(x+1)^{12}$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1+x)^{12} = 1 + 12x + \frac{12(11)}{2}x^2 + \frac{12(11)(10)}{6}x^3 + \dots$$

$$= \boxed{1 + 12x + 66x^2 + 220x^3 + \dots}$$

(b: #6) Find the first four terms of the binomial series for $(1 - \frac{x}{2})^{-2}$.

$$(1 + \frac{x}{2})^n = 1 + n\frac{x}{2} + \frac{n(n-1)}{2!}(\frac{x}{2})^2 + \frac{n(n-1)(n-2)}{3!}(\frac{x}{2})^3 + \dots$$

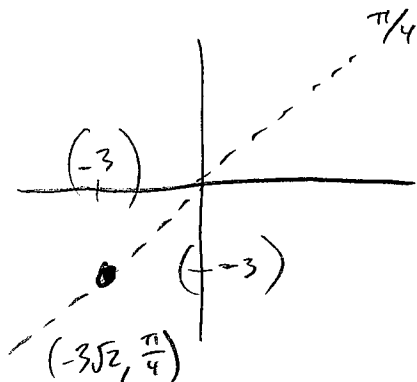
$$(1 + \frac{x}{2})^{-2} = 1 + ~~2~~\frac{x}{2} + \frac{-2(-3)}{2}(\frac{x}{2})^2 + \frac{-2(-3)(-4)}{6}(\frac{x}{2})^3 + \dots$$

$$= \boxed{1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3 + \dots}$$

(See 8.10 #1-14 for more examples concerning binomial series.)

5. Polar Coordinates (9.1)

(a.) Graph the polar coordinate $(-3\sqrt{2}, \frac{\pi}{4})$ and give three other equivalent polar coordinates for that point.



$(-3\sqrt{2}, \frac{\pi}{4})$
$(-3\sqrt{2}, \frac{9\pi}{4})$
$(3\sqrt{2}, \frac{5\pi}{4})$
$(3\sqrt{2}, -\frac{3\pi}{4})$

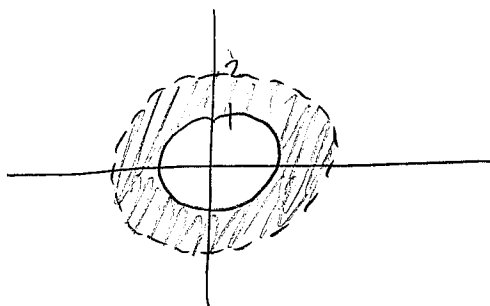
(b.) Give the Cartesian, that is, (x, y) coordinates for the polar coordinate $(-3\sqrt{2}, \frac{\pi}{4})$.

$$x = r \cos \theta = -3\sqrt{2} \cos\left(\frac{\pi}{4}\right) = -3\sqrt{2} \frac{\sqrt{2}}{2} = -3 \frac{2}{2} = -3$$

$$y = r \sin \theta = -3\sqrt{2} \sin\left(\frac{\pi}{4}\right) = -3\sqrt{2} \frac{\sqrt{2}}{2} = -3 \frac{2}{2} = -3$$

$(-3, -3)$

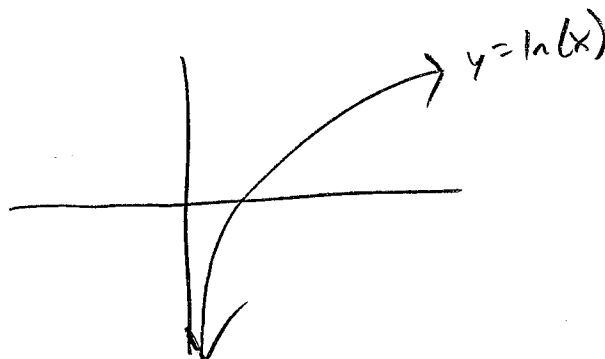
(c: #10) Graph the polar inequality $1 \leq r < 2$.



(d: #38) Graph the polar equation $r \sin \theta = \ln r + \ln \cos \theta$ by first changing it into a relation of x and y .

$$y = \ln(r \cos \theta)$$

$$y = \ln(x)$$



(See 9.1 #1-62 for more examples of using polar coordinates.)

6. Graphing in Polar Coordinates (9.2)

(a.) Find the slope of the line tangent to the polar graph of $\sqrt{2} + r \sec \theta = 0$ at the polar coordinate $(-1, \pi/4)$.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

$$= \frac{(1) \sin \pi/4 + (-1) \cos \pi/4}{(1) \cos \pi/4 - (-1) \sin \pi/4}$$

$$= \frac{\sqrt{2}/2 - \sqrt{2}/2}{\sqrt{2}/2 + \sqrt{2}/2} = \frac{0}{\sqrt{2}} = \boxed{0}$$

$$\sqrt{2} \cos \theta + r = 0$$

$$-\sqrt{2} \sin \theta + \frac{dr}{d\theta} = 0$$

$$-\sqrt{2} \left(\frac{\sqrt{2}}{2} \right) + \frac{dr}{d\theta} = 0$$

$$-1 + \frac{dr}{d\theta} = 0$$

$$\boxed{\frac{dr}{d\theta} = 1}$$

(b.) Show that the polar graph of $r + \sin(2\theta) = 2$ is rotationally symmetric about the origin.

$$(-r, \theta) \text{ or } (r, \pi + \theta)$$

~~$$\begin{array}{l} \text{LHS} \\ -r + \sin(-2\theta) \end{array}$$~~

~~$$\text{RHS} \\ 2$$~~

$$\begin{array}{l} \text{LHS} \\ r + \sin(2(\pi + \theta)) \end{array}$$

$$\begin{array}{l} \text{RHS} \\ 2 \end{array}$$

~~$$\begin{aligned} -r + \sin(-2\theta) &= -r - \sin 2\theta \\ &= -(r + \sin 2\theta) \\ &= -2 \end{aligned}$$~~

$$\begin{aligned} r + \sin(2(\pi + \theta)) &= r + \sin(2\pi + 2\theta) \\ &= r + \sin(2\theta) \\ &= 2 \quad \checkmark \end{aligned}$$

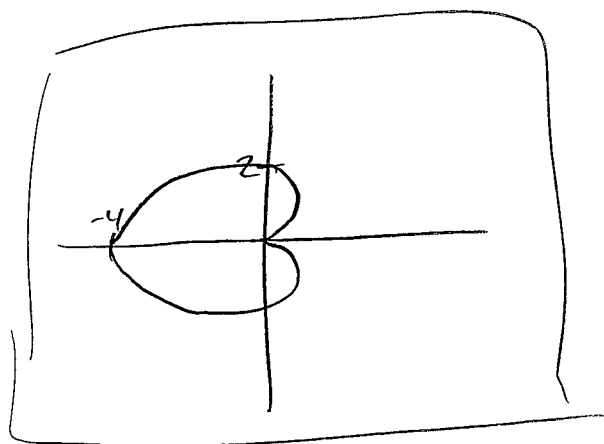
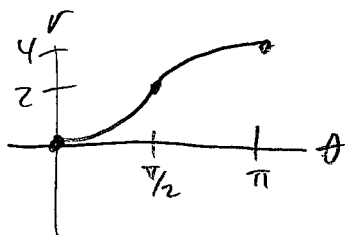
(c.) Show that the polar graph of $r = 2 - 2 \cos \theta$ is symmetric across the x-axis.

$$(r, -\theta) \text{ OR } (-r, \pi - \theta)$$

$$\frac{\text{LHS}}{r} \qquad \frac{\text{RHS}}{2 - 2 \cos(-\theta)}$$

$$2 - 2 \cos(-\theta) = 2 - 2 \cos \theta = r \quad \checkmark$$

(d.) Sketch the polar graph of $r = 2 - 2 \cos \theta$.



(See 9.2 #1-16 for more examples of graphing with polar curves.)

(Answers to (a) + (b) are reversed.)

7. Areas and Lengths in Polar Coordinates (9.3)

(a.) Find the length of the perimeter of the cardioid given by $r = 2 - 2 \cos \theta$.

$$\begin{aligned}
 A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta \\
 &= \int_0^{2\pi} \frac{1}{2} (2 - 2 \cos \theta)^2 d\theta \\
 &= \int_0^{2\pi} \frac{1}{2} (4 - 8 \cos \theta + 4 \cos^2 \theta) d\theta \\
 &= \int_0^{2\pi} 2 - 4 \cos \theta + 2 \cos^2 \theta d\theta \\
 &= \int_0^{2\pi} 2 - 4 \cos \theta + 2 \left(\frac{1 + \cos 2\theta}{2} \right) d\theta \\
 &= \int_0^{2\pi} 3 - 4 \cos \theta + \cos 2\theta d\theta \\
 &= \left[3\theta - 4 \sin \theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi}
 \end{aligned}$$

$$6\pi$$

(b.) Find the area inside of the cardioid given by $r = 2 - 2 \cos \theta$.

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad \frac{dr}{d\theta} = 2 \sin \theta$$

$$= \int_0^{2\pi} \sqrt{(4 - 8 \cos \theta + 4 \cos^2 \theta) + 4 \sin^2 \theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{4 - 8 \cos \theta + 4} d\theta$$

$$= \int_0^{2\pi} \sqrt{8 - 8 \cos \theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{16 \sin^2 \left(\frac{\theta}{2}\right)} d\theta$$

$$= \int_0^{2\pi} 4 \sin \left(\frac{\theta}{2}\right) d\theta$$

$$= \left[-8 \cos \left(\frac{\theta}{2}\right) \right]_0^{2\pi}$$

$$= -8 \cos \pi + 8 \cos 0$$

$$= 8 + 8$$

$$= 16$$

$$\begin{aligned}
 \sin^2 \theta &= \frac{1 - \cos 2\theta}{2} \\
 \sin^2 \frac{\theta}{2} &= \frac{1 - \cos \theta}{2} \\
 16 \sin^2 \frac{\theta}{2} &= 8 - 8 \cos \theta
 \end{aligned}$$

(See 9.3 #1-6, #17-25 for more examples of polar areas and curves.)