Calculus II Spring 2010 — Test 4

Your Name: Answer Key

Instructor: Steven Clontz

Draw a box around your final answer. You must show all work to receive credit.

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1. (10 pts) Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{\sqrt{n^3}}$.

Abs Ratio Test

$$\lim_{n\to\infty}\left|\frac{\frac{(4x-1)^{n+1}}{(n+1)^{3/2}}}{\frac{(4x-1)^n}{(n)^{3/2}}}\right| = \lim_{n\to\infty}\left|\frac{(4x-1)^{n+1}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{(4x-1)^n}\right| = \lim_{n\to\infty}\left|(4x-1)\left(\frac{n}{n+1}\right)^{3/2}\right|$$

$$= |4x-1| \lim_{n\to\infty}\left(\left|\frac{n}{n+1}\right|\right)^{3/2}$$

$$= |4x-1|$$

Converges for $|4x-1| < 1$

$-1 < 4x-1 < 1$

$0 < 4x < 2$

$\boxed{0 \le x \le 1/2}$

$x=0$: $\sum \frac{(-1)^n}{\sqrt{n^3}}$ Converges Absolutely

$x=1/2$: $\sum \frac{(2-1)^n}{\sqrt{n^3}} = \sum \frac{1}{n^{3/2}}$ Converges

Radius: $\boxed{1/4}$

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2. (10 pts) Find the Taylor Polynomial $P_2(x)$ of order 2 generated by $f(x) = \ln(x)$ at $x = 1$.

$f^{(0)}(x) = \ln x \qquad f^{(0)}(1) = 0$

$f^{(1)}(x) = \frac{1}{x} \qquad f^{(1)}(1) = 1$

$f^{(2)}(x) = -\frac{1}{x^2} \qquad f^{(2)}(1) = -1$

$$P_2(x) = \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}(x-1)^2$$

$$= \boxed{(x-1) - \tfrac{1}{2}(x-1)^2}$$

3. (10 pts) Find the Maclaurin Series generated by $f(x) = \dfrac{e^x + e^{-x}}{2}$. (This is known as the hyperbolic cosine function $\cosh(x)$, although that's not necessary to know to solve this problem.)

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!}(x)^{2k} + \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!}x^{2k+1}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}}$$

$$f^{(0)}(x) = \frac{e^x + e^{-x}}{2}$$

$$f^{(1)}(x) = \frac{e^x - e^{-x}}{2}$$

$$f^{(2)}(x) = \frac{e^x + e^{-x}}{2}$$

$\vdots$

$$f^{(2k)}(x) = \frac{e^x + e^{-x}}{2} \qquad f^{(2k+1)}(x) = \frac{e^x - e^{-x}}{2}$$

$$f^{(2k)}(0) = \frac{1+1}{2} = 1 \qquad f^{(2k+1)}(0) = \frac{1-1}{2} = 0$$

OR

$$f^{(k)}(x) = \frac{e^x + (-1)^k e^{-x}}{2}$$

$$f^{(k)}(0) = \frac{1 + (-1)^k}{2}$$

$$\sum_{k=0}^{\infty} \frac{\frac{1+(-1)^k}{2}}{k!} x^k = \boxed{\sum_{k=0}^{\infty} \frac{1+(-1)^k}{2(k!)} x^k}$$

4. (10 pts) Recall that $\cos(x) = \lim_{n\to\infty} P_n(x) + \lim_{n\to\infty} R_n(x)$ where $\lim_{n\to\infty} P_n(x)$ is its Maclaurin Series and $R_n(x)$ is the error/remainder term given by Taylor's Formula.

Show that the Maclaurin Series generated by $\cos(x)$

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$$

converges to $\cos(x)$ for any real number x by showing $\lim_{n\to\infty} R_n(x) = 0$.

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{f^{(n+1)}(c_n)}{(n+1)!} x^{n+1} \right| \le \lim_{n\to\infty} \left| \frac{1}{(n+1)!} x^{n+1} \right| = 0 \quad ✓$$

$$\left(\text{Note: } |f^{(n+1)}(c_n)| = |\cos(c_n)| \text{ or } |\sin(c_n)| \le 1\right)$$

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5. (10 pts) Express $\int e^{x^3}\,dx$ as a power series.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{x^3} = \sum_{k=0}^{\infty} \frac{x^{3k}}{k!}$$

$$\int e^{x^3} dx = \boxed{\sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)k!} + C}$$

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6. (10 pts) Give a polynomial of degree 6 which approximates $f(x) = \sqrt[3]{1+x^3}$.

$(1+x)^{1/3} \approx 1 + \frac{1}{3}x + \frac{(1/3)(-2/3)}{2}x^2$

$\approx 1 + \frac{1}{3}x - \frac{1}{9}x^2$

$(1+x^3)^{1/3} \approx \boxed{1 + \frac{1}{3}x^3 - \frac{1}{9}x^6}$

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7. (5 points total) For the polar point $(r, \theta) = (3, 3\pi/4)$,

- (1 pt) Give an equivalent polar point.

$\boxed{(-3, -\pi/4)}$ OR $\boxed{(3, 11\pi/4)}$ (etc.)

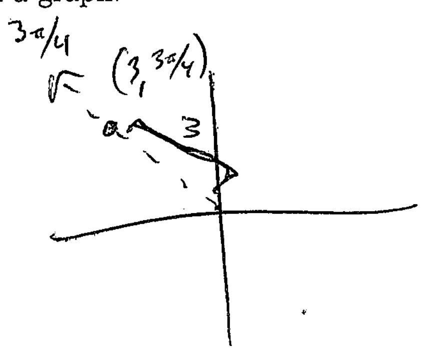
- (2 pts) Give the corresponding Cartesian (x, y) expression.

$x = r\cos\theta = 3\cos 3\pi/4 = 3(-\sqrt{2}/2) = -3\sqrt{2}/2$

$y = r\sin\theta = 3\sin 3\pi/4 = 3(\sqrt{2}/2) = 3\sqrt{2}/2$

$\boxed{\left(-3\sqrt{2}/2,\ 3\sqrt{2}/2\right)}$

- (2 pts) Plot it on a graph.



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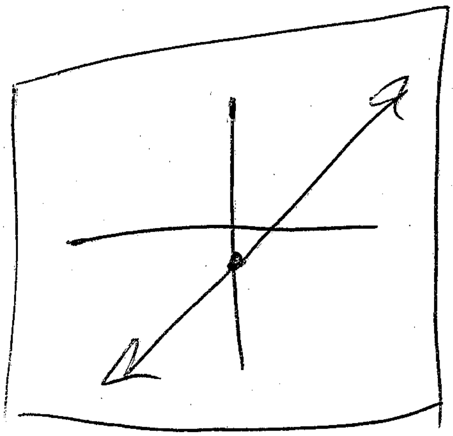
8. (10 pts) Sketch the graph of the polar equation $r \sin\theta + \sin^2\theta = r\cos\theta - \cos^2\theta$

$y + \sin^2\theta + \cos^2\theta = x$

$y + 1 = x$

$y = x - 1$



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9. (5 pts) Show that the polar graph of $r = 4 - 4\cos(\theta)$ is vertically symmetric across the x-axis.

x-axis $(r, -\theta)$ or $(-r, \pi - \theta)$

LHS	RHS
r	$4 - 4\cos(-\theta)$

$4 - 4\cos(-\theta) = 4 - 4\cos(\theta) = r$ ✓

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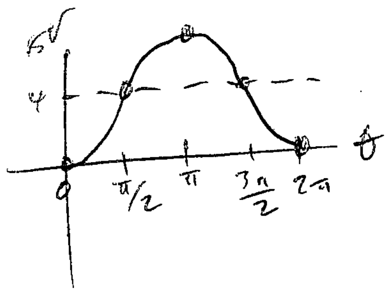
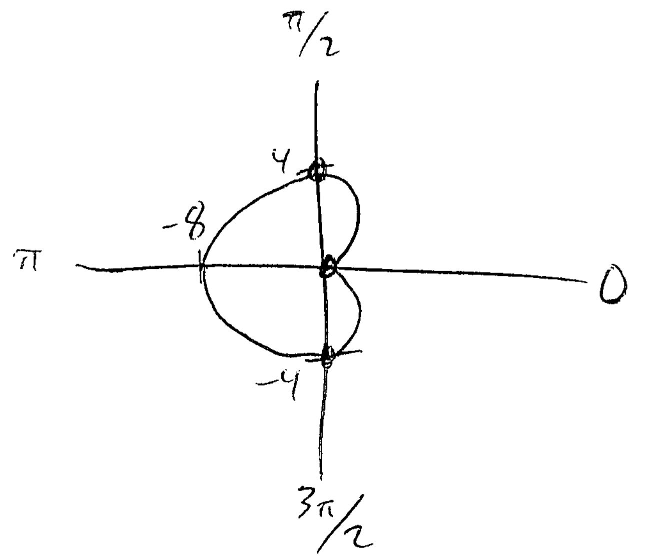
10. (10 pts) Sketch the polar graph of $r = 4 - 4\cos(\theta)$.





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11. (5 pts) Show that the area of the region bounded by the polar equation $r = 4 - 4\cos(\theta)$ is equal to the integral $\int_0^{2\pi} (8 - 16\cos\theta + 8\cos^2\theta)d\theta$. (If you compute the exact value for the areas, I'll award 3 bonus points.)

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$8\cos^2\theta = 4 + 4\cos 2\theta$

$$A = \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(4 - 4\cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(16 - 32\cos\theta + 16\cos^2\theta)\, d\theta$$

$$= \boxed{\int_0^{2\pi} \left[8 - 16\cos\theta + 8\cos^2\theta\right] d\theta}$$

Bonus

$$= \int_0^{2\pi} 8 - 16\cos\theta + 4 + 4\cos 2\theta \, d\theta$$

$$= \int_0^{2\pi} 12 - 16\cos\theta + 4\cos 2\theta \, d\theta$$

$$= \Big[12\theta - 16\sin\theta + 2\sin 2\theta\Big]_0^{2\pi}$$

$$= \boxed{24\pi}$$

12. (5 pts) Show that the length of the perimeter of the region bounded by the polar equation $r = 4 - 4\cos(\theta)$ is equal to the integral $\int_0^{2\pi} \sqrt{32 - 32\cos\theta}\, d\theta$. (If you compute the exact value for the length, I'll award 3 bonus points.)

$\frac{dr}{d\theta} = 4\sin\theta$

$L = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{16\sin^2\theta + (4 - 4\cos\theta)^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{16\sin^2\theta + 16 - 32\cos\theta + 16\cos^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{32 - 32\cos\theta}\, d\theta$

Bonus

~~$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$~~

~~$64\cos^2\frac{\theta}{2} = 32 + 32\cos\theta$~~

~~$= \int_0^{2\pi} \sqrt{64\cos^2\frac{\theta}{2}}\, d\theta$~~

~~$= \int_0^{2\pi} 8\left|\cos\frac{\theta}{2}\right| d\theta$~~

~~$= 2\int_0^{\pi} 8\cos\frac{\theta}{2}\, d\theta$~~

~~$= 2\left[16\sin\frac{\theta}{2}\right]_0^{\pi} = 2[16 - 0] = 32$~~

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13. (BONUS: 5 pts) Show that for $0 < x < 1$, $\sum_{n=0}^{\infty} x^n(2-x)^n = \sum_{n=0}^{\infty} nx^{n-1}$. (HINT: Use the Geometric Series $\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$ for $-1 < y < 1$ and term-by-term differentiation)

$|x| < 1$

$0 < x < 1$

$(0 < 2x < 2)$

$-(0 < x^2 < 1)$

$0 < 2x - x^2 < 1$

$|2x - x^2| < 1$

So $\sum_{n=0}^{\infty} x^n(2-x)^n$

$= \sum_{n=0}^{\infty} (2x - x^2)^n$

$= \frac{1}{1-(2x-x^2)}$

$= \frac{1}{1-2x+x^2}$

$= \frac{1}{(1-x)^2}$

$= \frac{d}{dx}\left[\frac{1}{1-x}\right]$

$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right]$

$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[x^n\right]$

$= \sum_{n=0}^{\infty} nx^{n-1}$ ✓

Incidentally, the original version of #5 isn't too much harder...

Original #5 Express [illegible] $\int x e^{x^3}\,dx$ as a power series.

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{x^3} = \sum_{k=0}^{\infty} \frac{x^{3k}}{k!}$$

$$x e^{x^3} = \sum_{k=0}^{\infty} x\, \frac{x^{3k}}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{x^{3k+1}}{k!}$$

$$\int x e^{x^3}\,dx = \boxed{\sum_{k=0}^{\infty} \frac{x^{3k+2}}{(3k+2)\,k!} + C}$$

And also...

$$x^2 e^{x^3} = \sum_{k=0}^{\infty} x^2\, \frac{x^{3k}}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{x^{3k+2}}{k!}$$

So

$$\int x^2 e^{x^3}\,dx = \sum_{k=0}^{\infty} \frac{x^{3k+3}}{(3k+3)\,k!} + C$$

$$= \sum_{k=0}^{\infty} \frac{(x^3)^{k+1}}{3(k+1)\,k!} + C$$

$$= \frac{1}{3} \sum_{k=0}^{\infty} \frac{(x^3)^{k+1}}{(k+1)!} + C$$

$$= \frac{1}{3} \sum_{k=1}^{\infty} \frac{(x^3)^{k}}{k!} + C$$

$$= \frac{1}{3} \sum_{k=1}^{\infty} \frac{(x^3)^{k}}{k!} + \frac{1}{3} + C$$

$$= \frac{1}{3} \sum_{k=1}^{\infty} \frac{(x^3)^{k}}{k!} + \frac{1}{3}\,\frac{(x^3)^0}{0!} + C$$

$$= \frac{1}{3} \sum_{k=0}^{\infty} \frac{(x^3)^{k}}{k!} + C$$

$$= \boxed{\frac{1}{3} e^{x^3} + C}$$