Calculus II Spring 2010 Test 1

Your Name: $\boxed{\text{Answer Key}}$ Instructor: Steven Clontz

Draw a box around your final answer. You must show all work to receive credit.

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1. (10 pts.) A three-dimensional solid runs from $x = -2$ to $x = 2$. The area of a cross-section perpendicular to the x-axis is given by the formula $A(x) = \cos(\frac{\pi}{4}x)$ for any x. Find the volume of this solid.

$$V = \int_{-2}^{2} \cos\left(\tfrac{\pi}{4}x\right)dx = \int_{x=-2}^{x=2} \cos(u)\,\tfrac{4}{\pi}\,du$$

Let $u = \frac{\pi}{4}x$

$du = \frac{\pi}{4}dx$

$\frac{4}{\pi}du = dx$

$$= \tfrac{4}{\pi}\Big[\sin u\Big]_{x=-2}^{x=2}$$

$$= \tfrac{4}{\pi}\Big[\sin \tfrac{\pi}{4}x\Big]_{-2}^{2}$$

$$= \tfrac{4}{\pi}\Big[\left(\sin \tfrac{\pi}{2}\right) - \left(\sin -\tfrac{\pi}{2}\right)\Big]$$

$$= \tfrac{4}{\pi}\Big[1 - (-1)\Big]$$

$$= \boxed{8/\pi}$$

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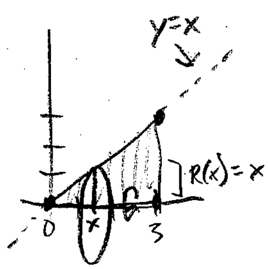
2. (10 pts.) Use the disk method to find the volume of the cone obtained by rotating a triangle with vertices at (3,0), (3,3), and (0,0) around the line ~~$y = 1$~~ $y=0$. (You can use the formula for the volume of a cone to check your answer, but you must use disk method correctly to earn full credit.)



$$V = \int_0^3 \pi R(x)^2\,dx$$

$$= \int_0^3 \pi x^2\,dx$$

$$= \tfrac{\pi}{3}\Big[x^3\Big]_0^3$$

$$= \tfrac{\pi}{3}\Big[27 \cancel{-0}\Big]$$

$$= \boxed{9\pi}$$

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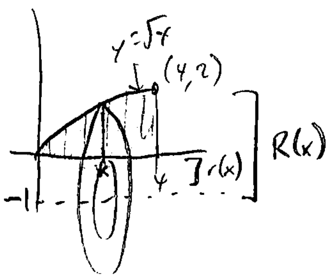
3. (15 pts.) Use the washer method to find the volume of the solid obtained by rotating the region bounded by $y = 0$, $y = \sqrt{x}$, and $x = 4$ around the line $y = -1$.



$R(x) = \sqrt{x} + 1$

$r(x) = 1$

$$V = \int_0^4 \pi \left(R(x)^2 - r(x)^2 \right) dx$$

$$= \int_0^4 \pi \left((\sqrt{x}+1)^2 - 1 \right) dx$$

$$= \int_0^4 \pi \left(x + 2x^{1/2} \cancel{+1-1} \right) dx$$

$$= \pi \left[\tfrac{1}{2} x^2 + \tfrac{4}{3} x^{3/2} \right]_0^4$$

$$= \pi \left[\left(\tfrac{1}{2}(16) + \tfrac{4}{3}(8) \right) - \cancel{(0+0)} \right]$$

$$= \pi \left[8 + \tfrac{32}{3} \right]$$

$$= \pi \left(\tfrac{24}{3} + \tfrac{32}{3} \right)$$

$$= \boxed{\tfrac{56}{3}\pi}$$

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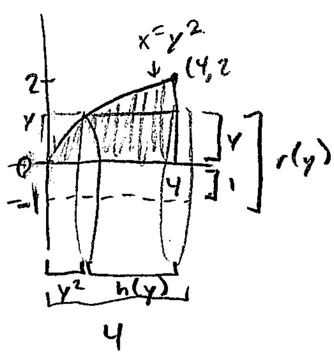
4. (15 pts.) Now use the cylindrical shell method to find the volume of the solid from question #3.



$y^2 + h(y) = 4$

$h(y) = 4 - y^2$

$r(y) = 1 + y$

$$V = \int_0^2 2\pi\, r(y)\, h(y)\, dy$$

$$= \int_0^2 2\pi (1+y)(4-y^2)\, dy$$

$$= \int_0^2 2\pi \left(4 + 4y - y^2 - y^3 \right) dy$$

$$= 2\pi \left[4y + 2y^2 - \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 \right]_0^2$$

$$= 2\pi \left[\left(4(2) + 2(4) - \tfrac{1}{3}(8) - \tfrac{1}{4}(16) \right) - \cancel{(0+0-0-0)} \right]$$

$$= 2\pi \left[8 + 8 - \tfrac{8}{3} - 4 \right]$$

$$= 2\pi \left(12 - \tfrac{8}{3} \right)$$

$$= 2\pi \left(\tfrac{36}{3} - \tfrac{8}{3} \right)$$

$$= 2\pi \left(\tfrac{28}{3} \right) = \boxed{\tfrac{56}{3}\pi}$$

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5. (15 pts.) Find the length of the curve given by the parametric equations $x = \cos(t^2 + \pi)$ and $y = \sin(t^2 + \pi)$ for $0 \le t \le \pi$.

$\frac{dx}{dt} = -2t \sin(t^2+\pi)$

$\frac{dy}{dt} = 2t\cos(t^2+\pi)$

$$s = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \sqrt{4t^2 \sin^2(t^2+\pi) + 4t^2\cos^2(t^2+\pi)}\, dt$$

$$= \int_0^{\pi} \sqrt{4t^2}\sqrt{\cancel{\sin^2(t^2+\pi) + \cos^2(t^2+\pi)}}\, dt$$

$$= \int_0^{\pi} 2t\, dt$$

$$= \left[t^2\right]_0^{\pi} = \boxed{\pi^2}$$

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6. (10 pts.) Find the length of the curve $\cancel{h(x)}\ y = \frac{1}{3}(2 + x^2)^{3/2} + \frac{1}{3}$ on the interval $0 \le x \le 3$.
(Hint: Recall $a^2 + 2ab + b^2 = (a + b)^2$.)

$$\frac{dy}{dx} = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}(2+x^2)^{1/2}(\cancel{2}x)$$

$$= x\sqrt{2+x^2}$$

$$s = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^3 \sqrt{1 + x^2(2+x^2)}\, dx$$

$$= \int_0^3 \sqrt{1+2x^2+x^4}\, dx$$

$$= \int_0^3 \sqrt{(1+x^2)^2}\, dx$$

$$= \int_0^3 1+x^2\, dx$$

$$= \left[x + \tfrac{1}{3}x^3\right]_0^3$$

$$= \left(3 + \tfrac{1}{3}(27)\right) - \cancel{(0+0)}$$

$$= 3+9$$

$$= \boxed{12}$$

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7. (15 pts.) Recall Hooke's Law, which says that the force required to stretch/compress a spring x from its natural length is given by $F(x) = kx$. A particular spring's natural length is 5 inches, and the spring's force constant k is 40 pounds per inch. How much work is done in compressing the spring to 2 inches?

$$W = \int_0^3 40x \, dx$$
$$= \left[20x^2\right]_0^3$$
$$= 20(9) - \cancel{20(0)}$$
$$= \boxed{180}$$

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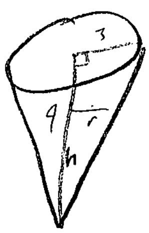
8. (10 pts.) A conical tank of height 9m and radius 3m stands on its point. A liquid weighing 10,000 N/m^3 is pumped into the tank from its point. How much work is done in filling the tank to a height of 3m?



$$\frac{3}{r} = \frac{9}{h}$$
$$3h = 9r$$
$$r = \tfrac{1}{3}h$$

$$V = \tfrac{1}{3}\pi r^2 h$$
$$V = \tfrac{1}{3}\pi \left(\tfrac{1}{3}h\right)^2 h$$
$$V = \tfrac{1}{27}\pi h^3$$
$$\frac{dV}{dh} = \tfrac{1}{9}\pi h^2$$
$$\frac{dF}{dh} = \frac{10000}{9}\pi h^2$$
$$\frac{dW}{dh} = \frac{10000}{9}\pi h^3$$

$$W = \int \frac{10000}{9}\pi h^3 \, dh$$
$$W = \frac{1}{4}\,\frac{10000}{9}\pi h^4$$
$$W(3) = \frac{1}{\cancel{4}}\,\frac{\overset{2500}{\cancel{10000}}}{\cancel{9}}\pi \overset{9}{\cancel{(81)}}$$
$$= \boxed{22500\pi}$$

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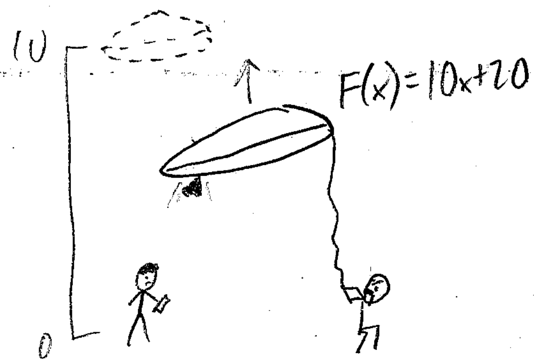
9. (BONUS - 5pts.) Balloon Boy is floating away! Sean Hannity shoots a blowdart into the balloon, which causes it to slowly leak air, while Wolf Blitzer manages to grab a rope trailing from the craft and pull it down to the ground from a height of 10 meters. If the leaking balloon exerts an upward force of $10x + 20$ N, where x is the height of the balloon in meters, find how much work Wolf did in pulling down the balloon.



$$W = \int_0^{10} 10x + 20 \, dx$$

$$= \left[5x^2 + 20x\right]_0^{10}$$

$$= (500 + 200) \cancel{-(0+0)}$$

$$= \boxed{700}$$

10. (BONUS - 1pt.) Hannity discovers that the balloon is empty! Where was Balloon Boy all along?

Hiding in his parents' attic "for the show".