Calculus II Spring 2010 Test 3a

Your Name: Answer Key Instructor: Steven Clontz

Draw a box around your final answer. You must show all work to receive credit.

---

1. (5 pts.) Find a formula (recursive or explicit) for $b_n = \left\langle \frac{1}{1+1}, -\frac{1}{4+1}, \frac{1}{9+1}, -\frac{1}{17}, \frac{1}{26}, -\frac{1}{37}, \dots \right\rangle$

$$\boxed{b_n = \frac{(-1)^{n+1}}{n^2+1}}$$

---

2. (5 pts.) Compute b_7 for the sequence defined in #1.

$$b_7 = \frac{\cancel{(-1)^{7+1}}}{7^2+1} = \boxed{\frac{1}{50}}$$

---

3. (10 pts.) Show that the sequence $a_n = \dfrac{n+1}{n^2}$ converges or diverges.

$$\lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$$

a_n $\boxed{\text{converges}}$ (to 0)

For problems 4-9, if the series converges, I'll award 5 BONUS points if you can give the exact value of the series. (Note: Some convergent series might be impossible to compute the value of using the techniques I've taught in class!) Make sure you label the value in the problem.

---

4. (15 pts.) Show that the series $\sum_{n=3}^{\infty} \dfrac{4^n}{(2n)!}$ converges or diverges.

Ratio Test

$$\lim_{n\to\infty} \frac{\frac{4^{n+1}}{(2(n+1))!}}{\frac{4^n}{(2n)!}} = \lim_{n\to\infty} \frac{4^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{4^n} = \lim_{n\to\infty} \frac{\cancel{4^n}\cdot 4}{\cancel{(2n)!}(2n+1)(2n+2)} \cdot \frac{\cancel{(2n)!}}{\cancel{4^n}}$$

$$= \lim_{n\to\infty} \frac{4}{(2n+1)(2n+2)} = 0 < 1$$

So $\sum_{n=3}^{\infty} \frac{4^n}{(2n)!}$ $\boxed{\text{converges}}$ by Ratio Test

---

5. (15 pts.) Show that the series $\sum_{i=1}^{\infty} \frac{2}{3^i} - \frac{2}{3^{i+1}}$ converges or diverges.

(Telescoping)

$$= \lim_{n\to\infty} \frac{2}{3} - \frac{2}{3^{n+1}} = \frac{2}{3}$$

The series converges to $\boxed{2/3}$ Bonus

---

6. (15 pts.) Show that the series $\sum_{k=2}^{\infty} \frac{k}{4+k^2}$ converges or diverges.

(LCT)

$$\lim_{n\to\infty} \frac{\frac{k}{4+k^2}}{\frac{1}{k}} = \lim_{n\to\infty} \frac{k^2}{4+k^2} = \lim_{n\to\infty} \frac{\cancel{k^2}}{\cancel{k^2}} = 1 \quad \& \quad \sum \frac{1}{k} \text{ diverges}$$

So by LCT $\sum_{k=2}^{\infty} \frac{k}{4+k^2}$ also diverges.

---

7. (10 pts.) Show that $\sum_{\cancel{m=1}\ m=2}^{\infty} \frac{1}{m \ln m}$ converges or diverges.

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x}\,dx$$

Let $u = \ln x$
$du = \frac{1}{x}dx$

$$= \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{1}{u}\,du$$

$$= \lim_{b\to\infty} \Big[\ln|u|\Big]_{x=2}^{x=b}$$

$$= \lim_{b\to\infty} \Big[\ln|\ln x|\Big]_2^b$$

$$= \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2)$$

$$= \infty$$

So $\sum_{m=2}^{\infty} \frac{1}{m \ln m}$ $\boxed{\text{diverges}}$ by the Integral Test.

---

8. (15 pts.) Show that the series $\sum_{j=0}^{\infty} \frac{\pi^j}{2(3^j)}$ converges or diverges.

$$= \sum_{j=0}^{\infty} \frac{1}{2}\left(\frac{\pi}{3}\right)^j = \sum_{j=1}^{\infty} \frac{1}{2}\left(\frac{\pi}{3}\right)^{j-1}$$

Geometric Series $\boxed{\text{diverges.}}$ as $\left|\frac{\pi}{3}\right| > 1$

---

9. (10 pts.) Show that $\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{2n}}$ converges absolutely, converges conditionally, or diverges.

Check $\sum_{n=0}^{\infty} \left| \frac{(-2)^n}{3^{2n}} \right| = \sum_{n=0}^{\infty} \frac{2^n}{3^{2n}}$

Root Test

$$\lim_{n\to\infty} \left(\frac{2^n}{3^{2n}}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{3^2} = \frac{2}{9} < 1$$

Thus $\sum_{n=0}^{\infty} \left| \frac{(-2)^n}{3^{2n}} \right|$ converges $+ \sum_{n=0}^{\infty} \frac{(-2)^n}{3^{2n}}$ $\boxed{\text{converges absolutely}}$

To find value...

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{2n}} = \sum_{n=0}^{\infty} \left(-\frac{2}{9}\right)^n = \sum_{n=1}^{\infty} (1)\left(-\frac{2}{9}\right)^{n-1} = \frac{1}{1+\frac{2}{9}} = \frac{1}{\frac{11}{9}} = \boxed{\frac{9}{11}}$$

BONUS

10. (BONUS: 3 pts.) Draw a smiley face here if you attended class on March 12. (Do NOT put anything here if you weren't in class. If you try to pull one over on me, I'll give you a zero for the test.)

11. (BONUS: 5 pts.) Fill in the blank areas to help me prove that $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges!

Apply the Ratio Test with $a_n = \frac{n!}{n^n}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}}{\cancel{(n+1)^{n+1}}} \cdot \frac{n^n}{\cancel{n!}}$$

$$\cancel{n!(n+1)} \qquad (n+1)^n\cancel{(n+1)}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

Let $L_n = \left(\frac{n}{n+1}\right)^n$.

$$\ln L_n = n \ln\left(\frac{n}{n+1}\right)$$

$$= \frac{\ln\left(\frac{n}{n+1}\right)}{1/n}$$

$$= \frac{\ln(n) - \ln(n+1)}{1/n}$$

$$\lim_{n\to\infty} \ln L_n = \ln(\lim_{n\to\infty} L_n) = \lim_{n\to\infty} \frac{\ln(n) - \ln(n+1)}{1/n}$$

(You may use LH here since the numerator can be rewritten to go to 0 instead of $\infty - \infty$.)
(Also remember $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.)

$$\ln(\lim_{n\to\infty} L_n) \overset{LH}{=} \lim_{n\to\infty} \frac{1/n - 1/(n+1)}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{1/n(n+1)}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+n}$$

$$= \lim_{n\to\infty} -\frac{\cancel{n^2}}{\cancel{n^2}}$$

$$= -1$$

Since $\ln(\lim_{n\to\infty} L_n) = -1$, $\lim_{n\to\infty} L_n = \frac{1}{e} < 1$, and $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the Ratio Test!

Calculus II Spring 2010 **Test 3b**

Your Name: Answer Key **Instructor: Steven Clontz**

Draw a box around your final answer. You must show all work to receive credit.

---

1. (5 pts.) Find a formula (recursive or explicit) for $b_n = \left\langle -\frac{1}{1+2}, \frac{1}{4+2}, -\frac{1}{9+2}, \frac{1}{18}, -\frac{1}{27}, \frac{1}{38}, \ldots \right\rangle$

$$\boxed{b_n = \frac{(-1)^n}{n^2+2}}$$

---

2. (5 pts.) Compute b_7 for the sequence defined in #1.

$$b_7 = \frac{(-1)^7}{7^2+2} = \boxed{-\frac{1}{51}}$$

---

3. (10 pts.) Show that the sequence $a_n = \dfrac{n}{n^2+3}$ converges or diverges.

$$\lim_{n\to\infty} \frac{n}{n^2+3} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$$

a_n $\boxed{\text{Converges}}$ (to 0)

For problems 4-9, if the series converges, I'll award 5 BONUS points if you can give the exact value of the series. (Note: Some convergent series might be impossible to compute the value of using the techniques I've taught in class!) Make sure you label the value in the problem.

---

4. (15 pts.) Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$ converges or diverges.

Ratio Test

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(2(n+1)!)}}{\frac{2^n}{(2n)!}} = \lim_{n\to\infty} \frac{\cancel{2^{n+1}}}{\cancel{(2n+2)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{2^n}} = \lim_{n\to\infty} \frac{2}{(2n+1)(2n+2)} = 0 < 1$$

$\cancel{2^n} \cdot 2$

$\cancel{(2n)!}\,(2n+1)(2n+2)$

So $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$ $\boxed{\text{Converges}}$ by Ratio Test.

---

5. (15 pts.) Show that the series $\sum_{i=1}^{\infty} \frac{3}{4^i} - \frac{3}{4^{i+1}}$ converges or diverges.

(Telescoping)

$$= \lim_{n\to\infty} \frac{3}{4} - \frac{3}{4^{n+1}} = \frac{3}{4}$$

So $\sum_{i=1}^{\infty} \frac{3}{4^i} - \frac{3}{4^{i+1}}$ $\boxed{\text{converges}}$ to $\boxed{3/4}$ ← BONUS

---

6. (15 pts.) Show that the series $\sum_{k=3}^{\infty} \frac{k}{k^2-4}$ converges or diverges.

(LCT)

- $$\lim_{k\to\infty} \frac{\frac{k}{k^2-4}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2}{k^2-4} = \lim_{k\to\infty} \frac{\cancel{k^2}}{\cancel{k^2}} = 1$$
- $\sum \frac{1}{k}$ diverges

So $\sum_{k=3}^{\infty} \frac{k}{k^2-4}$ also $\boxed{\text{diverges}}$ by LCT

---

7. (10 pts.) Show that $\sum_{\substack{\cancel{m=1} \\ m=2}}^{\infty} \frac{3}{m \ln m}$ converges or diverges.

$$\int_2^\infty \frac{3}{x\ln x}\,dx = \lim_{b\to\infty}\int_2^b \frac{3}{x\ln x}\,dx$$

Let $u=\ln x$
$du=\frac{1}{x}dx$

$$= \lim_{b\to\infty}\int_{x=2}^{x=b}\frac{1}{u}\,du$$

$$= \lim_{b\to\infty}\Big[\ln|u|\Big]_{x=2}^{x=b}$$

$$= \lim_{b\to\infty}\Big[\ln|\ln x|\Big]_2^b$$

$$= \lim_{b\to\infty}\Big[\ln|\ln b| - \ln(\ln 2)\Big]$$

$$= \infty$$

So $\sum_{m=2}^{\infty}\frac{3}{m\ln m}$ also diverges

by Integral Test

---

8. (15 pts.) Show that the series $\sum_{j=0}^{\infty} \frac{e^j}{3(2^j)}$ converges or diverges.

$$= \sum_{j=0}^{\infty}\frac{1}{3}\left(\frac{e}{2}\right)^j = \sum_{j=1}^{\infty}\frac{1}{3}\left(\frac{e}{2}\right)^{j-1}$$

Geometric series with $\left|\frac{e}{2}\right| > 1$ diverges.

9. (10 pts.) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{2^{2n}}$ converges absolutely, converges conditionally, or diverges.

Show $\displaystyle\sum_{n=0}^{\infty} \left|\frac{(-3)^n}{2^{2n}}\right| = \sum_{n=0}^{\infty} \frac{3^n}{2^{2n}}$ converges:

Root Test:

$$\lim_{n\to\infty} \left(\frac{3^n}{2^{2n}}\right)^{1/n} = \lim_{n\to\infty} \frac{3}{2^2} = \frac{3}{4} < 1$$

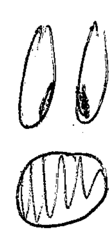
So $\displaystyle\sum_{n=0}^{\infty} \left|\frac{(-3)^n}{2^{2n}}\right|$ converges $+$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{2^{2n}}$ $\boxed{\text{converges absolutely}}$

For bonus:

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{2^{2n}} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} (1)\left(-\frac{3}{4}\right)^{n-1} = \frac{1}{1+\frac{3}{4}} = \frac{1}{\frac{7}{4}} = \boxed{\frac{4}{7}}$$

BONUS

10. (BONUS: 3 pts.) Draw a smiley face here if you attended class on March 12. (Do NOT put anything here if you weren't in class. If you try to pull one over on me, I'll give you a zero for the test.)



11. (BONUS: 5 pts.) Fill in the blank areas to help me prove that $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges!

Apply the Ratio Test with $a_n = \frac{n^n}{n!}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{\cancel{(n+1)^{n+1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$(n+1)^n \cancel{(n+1)} \qquad \cancel{n!}\ \cancel{(n+1)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

Let $L_n = \left(\frac{n+1}{n}\right)^n$.

$$\ln L_n = n \ln\left(\frac{n+1}{n}\right)$$

$$= \frac{\ln\left(\frac{n+1}{n}\right)}{1/n}$$

$$= \frac{\ln(n+1) - \ln(n)}{1/n}$$

$$\lim_{n\to\infty} \ln L_n = \ln(\lim_{n\to\infty} L_n) = \lim_{n\to\infty} \frac{\ln(n+1) - \ln(n)}{1/n}$$

(You may use LH here since the numerator can be rewritten to go to 0 instead of $\infty - \infty$.)
(Also remember $\frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)}$.)

$$\ln(\lim_{n\to\infty} L_n) \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{1/(n+1) - 1/n}{-1/n^2} = \lim_{n\to\infty} \frac{-\frac{1}{n(n+1)}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$$

Since $\ln(\lim_{n\to\infty} L_n) = 1$, $\lim_{n\to\infty} L_n = e > 1$, and $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by the Ratio Test!