Helpful Trig integrals to memorize:

$$\int \sec\theta \, d\theta = \int \sec\theta \, \overbrace{\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}}^{=1} \, d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \, d\theta$$

Let $u = \sec\theta + \tan\theta$

$$du = \sec\theta\tan\theta + \sec^2\theta \, d\theta$$

$$du = \sec^2\theta + \sec\theta\tan\theta \, d\theta$$

$$= \int \frac{du}{u} = \ln|u| + C$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\int \tan\theta \, d\theta = \int \frac{\sin\theta}{\cos\theta} \, d\theta$$

Let $u = \cos\theta$

$du = -\sin\theta \, d\theta$

$-du = \sin\theta \, d\theta$

$$= \int \frac{-du}{u} = -\ln|u| + C$$

$$= -\ln|\cos\theta| + C$$

$$= \ln\left|(\cos\theta)^{-1}\right| + C$$

$$= \ln|\sec\theta| + C$$

7.3 Trig Substitution

$$\int \frac{x^2}{\sqrt{4x^2-1}}\,dx$$

$\sec^2\theta - 1 = \tan^2\theta$

must match

Let $4x^2 = \sec^2\theta$

$2x = \sec\theta$

$x = \frac{1}{2}\sec\theta$

$dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\frac{1}{4}\sec^2\theta}{\sqrt{\sec^2\theta-1}}\,\frac{1}{2}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\frac{1}{4}\sec^2\theta}{\cancel{\tan\theta}}\,\frac{1}{2}\sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \int \frac{1}{8}\sec^3\theta\,d\theta$$

$$= \frac{1}{8}\int \sec^2\theta\sec\theta\,d\theta$$

$$= \frac{1}{8}\int (1+\tan^2\theta)\sec\theta\,d\theta$$

$$= \frac{1}{8}\int \sec\theta + \sec\theta\tan^2\theta\,d\theta$$

$$= \frac{1}{8}\int \sec\theta\,d\theta + \frac{1}{8}\int \sec\theta\tan^2\theta\,d\theta$$

Let $u = \tan\theta \quad v' = \sec\theta\tan\theta$

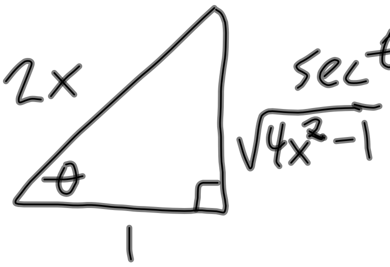
$u' = \sec^2\theta \quad v = \sec\theta$

$$= \frac{1}{8}\ln|\sec\theta+\tan\theta| + \frac{1}{8}\left(\sec\theta\tan\theta - \int \sec^3\theta\,d\theta\right)$$

$$\int \frac{1}{8}\sec^3\theta\,d\theta = \frac{1}{8}\ln|\sec\theta+\tan\theta| + \frac{1}{8}\sec\theta\tan\theta - \int \frac{1}{8}\sec^3\theta\,d\theta$$

$$2\int \frac{1}{8}\sec^3\theta\,d\theta = \frac{1}{8}\ln|\sec\theta+\tan\theta| + \frac{1}{8}\sec\theta\tan\theta + C$$

$$\int \frac{1}{8}\sec^3\theta\,d\theta = \frac{1}{16}\ln|\sec\theta+\tan\theta| + \frac{1}{16}\sec\theta\tan\theta + C$$



$\sec\theta = \frac{2x}{1} = \frac{hyp}{adj}$

$\tan\theta = \frac{opp}{adj} = \sqrt{4x^2-1}$

$$= \boxed{\frac{1}{16}\ln\left|2x+\sqrt{4x^2-1}\right| + \frac{1}{8}\,x\left(\sqrt{4x^2-1}\right) + C}$$

$$\int \frac{2x}{9x^2+16}\,dx \longleftarrow 16\tan^2\theta + 16 = 16\sec^2\theta$$

Let $9x^2 = 16\tan^2\theta$

$3x = 4\tan\theta$

$x = \frac{4}{3}\tan\theta$

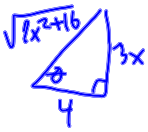
$dx = \frac{4}{3}\sec^2\theta\,d\theta$

$$= \int \frac{\frac{8}{3}\tan\theta}{16\tan^2\theta + 16}\,\frac{4}{3}\sec^2\theta\,d\theta$$

$$= \int \frac{\cancelto{2}{\frac{32}{9}}\tan\theta\,\cancel{\sec^2\theta}}{\cancel{16}\,\cancel{\sec^2\theta}}\,d\theta$$

$$= \int \frac{2}{9}\tan\theta\,d\theta$$

$$= \frac{2}{9}\ln|\sec\theta| + C$$



$\tan\theta = \frac{3x}{4} = \frac{opp}{adj}$

$\sec\theta = \frac{hyp}{adj} = \sqrt{9x^2+16}\,/4$

$$= \frac{2}{9}\ln\left|\frac{\sqrt{9x^2+16}}{4}\right| + C$$

$$= \frac{1}{9}\ln\left|\frac{9x^2+16}{16}\right| + C$$

$$= \frac{1}{9}\ln|9x^2+16| \cancel{- \frac{1}{9}\ln|16|} + C$$

$$= \frac{1}{9}\ln|9x^2+16| + C$$

$$\int \frac{2x}{9x^2+16}\,dx$$

Let $u = 9x^2 + 16$

$du = 18x\,dx$

$\frac{1}{9}du = 2x\,dx$

$$= \int \frac{\frac{1}{9}du}{u} = \frac{1}{9}\ln|u| + C = \boxed{\frac{1}{9}\ln|9x^2+16| + C}$$