Trig Integrals, continued (7.2)

Using Half-Angle Identities

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \sin^2(4x) = \frac{1-\cos(8x)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

These come from the Angle Sum formulas

Like

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(2\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$\int \sin^m x \cos^n x \, dx$ for m, n even:

Ex! $\int \sin^2 x \cos^2 x \, dx$

$$\boxed{\begin{aligned} \sin^2\theta &= \frac{1-\cos 2\theta}{2} \\ \cos^2\theta &= \frac{1+\cos(2\theta)}{2} \end{aligned}}$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \tfrac{1}{4} \int (1-\cos 2x)(1+\cos 2x) \, dx$$

$$= \tfrac{1}{4} \int 1 - \cancel{\cos 2x} \cancel{+\cos 2x} - \cos^2 2x \, dx$$

$$= \tfrac{1}{4} \int 1 - \cos^2 2x \, dx$$

$$= \tfrac{1}{4} \int \sin^2 2x \, dx$$

$$= \tfrac{1}{4} \int \frac{1-\cos 4x}{2} \, dx = \tfrac{1}{8} \int 1 - \cos 4x \, dx$$

$$\cancel{= \tfrac{1}{4} \int \tfrac{1}{2} - \tfrac{1}{2}\cos 4x \, dx}$$

$$\cancel{= \tfrac{1}{4}\left(\tfrac{1}{2}x\right) - \int \tfrac{1}{2}\cos 4x \, dx}$$

(Factored out ½ instead.)

$$= \tfrac{1}{8}\left(x - \int \cos 4x \, dx\right)$$

$$= \tfrac{1}{8}\left(x - \tfrac{1}{4}\sin 4x\right) + C$$

$$= \boxed{\tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C}$$

$$\int \sin^4 x \cos^4 x \, dx$$

$$= \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{16} \int (1-\cos 2x)^2 (1+\cos 2x)^2 \, dx$$

Foil both
of these

We can also use half-angle identities to get rid of roots.

$$\int \sqrt{1-\cos(4x)}\,dx \qquad \sin^2 2x = \frac{1-\cos 4x}{2}$$

$$2\sin^2 2x = 1-\cos 4x$$

$$= \int \sqrt{2\sin^2 2x}\,dx$$

$$= \int \sqrt{2}\sin 2x\,dx$$

$$= \sqrt{2}\left(-\tfrac{1}{2}\cos 2x\right)+C = \boxed{-\tfrac{\sqrt{2}}{2}\cos(2x)+C}$$

Roots don't always mean half-angle identities, though..

$$1 - \cos^2\theta = \sin^2\theta$$

$$\int \sqrt{1 - \cos^2 4x}\, dx$$

$$= \int \cancel{\sqrt{\vphantom{x}}}\, \sin^{\cancel{2}} 4x\, dx$$

$$= \int \sin 4x\, dx$$

$$= \boxed{-\tfrac{1}{4}\cos 4x + C}$$

One more type of identity: Product Identities

$$\cos(mx)\cos(nx) = \frac{\cos([m-n]x) + \cos([m+n]x)}{2}$$

$$\sin(mx)\sin(nx) = \frac{\cos([m-n]x) - \cos([m+n]x)}{2}$$

$$\sin(mx)\cos(nx) = \frac{\sin([-m-n]x) - \sin([m+n]x)}{2}$$

I don't expect you to memorize these. I'll give them to you on a test.

Also, any integral that requires these formulas could also be solved using cycling integration by parts anyway.

Example:

$$\boxed{\sin(mx)\sin(nx) = \frac{\cos([m-n]x) - \cos([m+n]x)}{2}}$$

$$\int \sin(1x)\sin(\tfrac{1}{2}x)\,dx$$

$$= \int \frac{\cos(\tfrac{1}{2}x) - \cos(\tfrac{3}{2}x)}{2}\,dx$$

$$= \int \tfrac{1}{2}\cos(\tfrac{1}{2}x) - \tfrac{1}{2}\cos(\tfrac{3}{2}x)\,dx$$

$$= \cancel{\tfrac{1}{2}}(\cancel{2})\sin(\tfrac{1}{2}x) - \cancel{\tfrac{1}{2}}(\cancel{2}/3)\sin(\tfrac{3}{2}x) + C$$

$$= \sin(\tfrac{1}{2}x) - \tfrac{1}{3}\sin(\tfrac{3}{2}x) + C$$

$$\cos(mx)\cos(nx) = \frac{\cos([m-n]x) + \cos([m+n]x)}{2}$$

$$\int \cos(\pi y)\cos(2\pi y)\,dy$$

$$= \int \frac{\cos(\pi y) + \cos(3\pi y)}{2}\,dy$$

$$= \int \tfrac{1}{2}\cos(\pi y) + \tfrac{1}{2}\cos(3\pi y)\,dy$$

$$= \tfrac{1}{2}\,\tfrac{1}{\pi}\sin(\pi y) + \tfrac{1}{2}\,\tfrac{1}{3\pi}\sin(3\pi y) + C$$

$$= \boxed{\frac{1}{2\pi}\sin(\pi y) + \frac{1}{6\pi}\sin(3\pi y) + C}$$