Final note from 8.7: Radius of Convergence

Find the interval of convergence and radius of convergence of the following series

#3 $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$

Abs Ratio Test

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}(4x+1)^{n+1}}{\cancel{(-1)^n}(4x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(4x+1)^n}(4x+1)}{\cancel{(4x+1)^n}} \right|$$

$$= \lim_{n\to\infty} |(4x+1)|$$

$$= |4x+1|$$

Int of convergence

$$|4x+1|<1 \rightarrow -1<4x+1<1$$

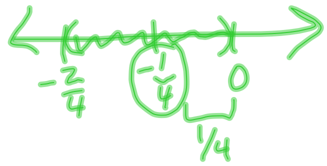
- Center of conv.

$$\rightarrow -\frac{1}{4} < x + \frac{1}{4} < \frac{1}{4}$$

Radius of convergence

$$\rightarrow -\frac{1}{4}-\frac{1}{4} < x < -\frac{1}{4}+\frac{1}{4}$$

$$-\frac{2}{4} < x < 0$$



Test $x=-\frac{1}{2}$, $x=0$

$\boxed{x=-\frac{1}{2}}$

#3 $\sum_{n=0}^{\infty} (-1)^n (4(-\frac{1}{2})+1)^n$

$= \sum_{n=0}^{\infty} (-1)^n (-1)^n$

$= \sum_{n=0}^{\infty} (\cancel{-1})^{2n}$ 1

diverges

$\boxed{x=0}$

$\sum_{n=0}^{\infty} (-1)^n \cancel{(4(0)+1)^n}$

$\sum_{n=0}^{\infty} (-1)^n$

Geo. Series with $|r|=|-1|\geq 1$ diverges

Int of Conv: $\boxed{-\frac{1}{2}<x<0}$ OR $\boxed{(-\frac{1}{2},0)}$

Rad. of Conv: $\boxed{\frac{1}{4}}$

Note: we can use $\sum_{n=0}^{\infty}(x)^n = \frac{1}{1-x}$ for $(-1,1)$ to get series representations for lots of functions:

Find a series representation for

$$f(x) = \frac{1}{1+x^2}$$

and state its interval & radius of convergence.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1\cdot x^2)^n$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n x^{2n}}$$

$$= \cancel{(-1)^0} x^{2\cdot 0} + (-1)^1 x^{2\cdot 1} + \cancel{(-1)^2} x^{2\cdot 2} + \cdots$$

$$= \boxed{1 - x^2 + x^4 - x^6 + x^8 - \cdots}$$

Find Int of Conv.

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}} x^{2(n+1)}}{\cancel{(-1)^n} x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\right| = \lim_{n\to\infty}|x^2|$$

$$= |x^2| = x^2$$

Converges for $x^2 < 1$ Radius of Conv.

$-1 < x < 1$

Test $\boxed{x=-1}$

$\sum_{n=0}^{\infty}(-1)^n \cancel{(-1)^{2n}}$

diverge

$\boxed{x=1}$

$\sum_{n=0}^{\infty}(-1)^n \cancel{(1)^{2n}}$

diverge

Int of Conv. $\boxed{(-1,1)}$ OR $\boxed{-1<x<1}$

Radius of Conv. $\boxed{1}$

Find a series representation for

$g(x) = \operatorname{Arctan}(x)$

and state its interval + radius of convergence.

From prev. example

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{Int of Conv: } (-1, 1)$$

$$\int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx \quad \text{Int of Conv: } (-1, 1)$$

$$\operatorname{Arctan}(x) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}\right) + C$$

$$\operatorname{Arctan}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \;\cancel{+C}$$

Find C

$$\operatorname{Arctan}(0) = \sum_{n=0}^{\infty} (-1)^n \frac{0^{\cancel{2n+1}}}{2n+1} + C$$

$$= \cancel{\sum_{n=0}^{\infty}} 0 + C$$

$$\operatorname{Arctan}(0) = C$$

$$0 = C$$

Notice:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

↓ Integral   ↓ Integral

$$\operatorname{Arctan} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

A more "straightforward" way of generating a series representation for a function:

A __Taylor Series__ generated by $f(x)$ at $x=a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

the k^{th} derivative of $f(x)$

Ex: $f^{(0)}(x) = f(x)$

$f^{(1)}(x) = f'(x)$

$\vdots$

$f^{(4)}(x) = f''''(x)$

$\vdots$

A __Maclaurin Series__ generated by $f(x)$ is the Taylor series when $a=0$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\cancel{(x-0)^k}\,x^k$$

It'll turn out that a Taylor or Maclaurin series often converges to the f(x) it was generated from.

Find the Maclaurin Series for f(x) = x^2+1

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \frac{f(0)}{\cancel{0!}} \cancel{x^0} + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \cdots$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f^{(3)}(0)}{6} x^3 + \cdots$$

$f(x) = x^2+1$

$f'(x) = 2x$

$f''(x) = 2$

$f'''(x) = 0$

$f^{(4)}(x) = 0$

$\vdots$

$$= 1 + \cancel{(0)x} + \frac{2}{2} x^2 + \cancel{0 + 0 \cdots}$$

$$= 1 + x^2$$