$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$$

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series w/ positive terms.

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges
- (If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ then we know nothing about $\sum_{n=1}^{\infty} a_n$!)

(Easy Way)

$$= \sum_{n=0}^{\infty} \frac{2^n}{3^n} + \sum_{n=0}^{\infty} \frac{5}{3^n}$$

$$= \sum_{n=1}^{\infty} \left(\tfrac{2}{3}\right)^{n-1} + \sum_{n=1}^{\infty} 5\left(\tfrac{1}{3}\right)^{n-1} = \frac{1}{1-\frac{2}{3}} + \frac{5}{1-\frac{1}{3}} = 3 + \frac{15}{2} = \boxed{\frac{21}{2}}$$

(Ratio Test)

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}+5}{3^{n+1}}}{\frac{2^n+5}{3^n}} = \lim_{n\to\infty} \frac{2^{n+1}+5}{3^{n+1}} \cdot \frac{3^n}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}+5}{3(\cancel{3^n})} \cdot \frac{\cancel{3^n}}{2^n+5}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}+5}{3(2^n+5)}$$

$$\frac{d}{dx}\left[2^{x+1}\right] = 2^{x+1}\ln(2)$$

$$\frac{d}{dx}\left[a^x\right] = a^x(\ln a)$$

$$\overset{LH}{=} \lim_{n\to\infty} \frac{2\,\cancel{2^{n+1}}\,\cancel{(\ln 2)}}{3(\cancel{2^n}\,\cancel{\ln(2)})}$$

$$= \tfrac{2}{3} < 1$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, so $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ converges

$$\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$$

$$\lim_{n\to\infty} \frac{\dfrac{4^{n+1}(n+1)!(n+1)!}{(2(n+1))!}}{\dfrac{4^n n! n!}{(2n)!}} = \lim_{n\to\infty} \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!} \frac{(2n)!}{4^n n! n!}$$

$$= \lim_{n\to\infty} \frac{4\cdot \cancel{4^n}\cancel{(n!)}(n+1)\cancel{(n!)}(n+1)}{\cancel{(2n)!}(2n+1)(2n+2)} \frac{\cancel{(2n)!}}{\cancel{4^n}\cancel{n!}\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{4(n+1)(n+1)}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{4n^2+8n+4}{4n^2+6n+2}$$

$$= 1$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{4}^{2}(n+1)\cancel{(n+1)}}{(2n+1)\cancel{(2n+2)}_{\cancel{2}}} = \frac{2(n+1)}{(2n+1)} = \frac{2n+2}{2n+1}$$

So $a_{n+1} > a_n$, so $\langle a_n \rangle$ is increasing.

$\Rightarrow \langle a_n \rangle \not\to 0$

$\Rightarrow \sum a_n$ diverges

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with (eventually) all nonnegative terms.

- If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges
- If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

(If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, we know nothing new.)

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Let $\sum_{n=1}^{\infty} a_n$ be a series with (eventually) all nonnegative terms.

- If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$ then $\sum_{n=1}^{\infty} a_n$ converges
- If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges
- (If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, we know nothing new.)

Easy way:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{1/2}{1-1/2} = 1$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{2^n}} = \lim_{n\to\infty} \frac{1}{(2^{\cancel{n}})^{\cancel{1/n}}} = \frac{1}{2} < 1$$

So $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by Root Test.

Useful fact: $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{(1)^2}{2} = \frac{1}{2} < 1$$

So $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges by Root Test.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{\sqrt[n]{2^n}}{\sqrt[n]{n^2}} = \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^2} = \frac{2}{(1)^2} = 2 > 1$$

So $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges by Root Test.

$$\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n = \frac{1}{2}+\left(\frac{1}{3}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{5}\right)^4+\cdots$$

$$\lim_{n\to\infty}\sqrt[\cancel{n}]{\left(\frac{1}{1+n}\right)^{\cancel{n}}} = \lim_{n\to\infty}\frac{1}{1+n} = 0 < 1$$

By the Root Test, $\sum_{n=1}^{\infty}\left(\frac{1}{1+n}\right)^n$ converges.