Absolute Ratio Test

Let $\sum_{n=0}^{\infty} a_n$ be a series of nonzero terms

- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum a_n$ converges absolutely.
- If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum a_n$ diverges.

How to find the interval of convergence for a power series $\sum_{n=0}^{\infty} c_n x^n$

1. Use Abs Ratio Test to find the values of x for which the series converges absolutely $\left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1\right)$ and the values for which the series diverges $\left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1\right)$
2. For the x values which make $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, determine convergence "by hand": plug in for x and use convergence tests.

Find the interval of convergence for

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

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① Use Abs Ratio Test to find the values of x for which the series converges absolutely $\left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1\right)$ and the values for which the series diverges $\left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1\right)$

② For the x values which make $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, determine convergence "by hand": plug in for x and use convergence tests.

Abs Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^n}\frac{x^{n+1}}{n+1}}{\cancel{(-1)^{n-1}}\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{\,x}}{n+1}\cdot\frac{n}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|x\,\frac{n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|$$

Converges for $|x|<1$

Diverges for $|x|>1$

Check $|x|=1 \to x=1$ or $x=-1$

$\boxed{x=1}$ ← converges

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(1)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

(Could use AST, but this is Alt. Harmonic Series, which converges)

$\boxed{x=-1}$ ← diverges

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}(-1)^{\overset{\text{odd}}{2n-1}}\frac{1}{n} = \sum_{n=1}^{\infty}-\frac{1}{n}$$

diverges (by p-series)

Interval of Convergence: $-1 < x \le 1$

$(-1, 1]$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x+1)^{2n-1}}{2n-1}$$

$$\left(\frac{(x+1)^{2(n+1)-1}}{2(n+1)-1} \right)$$

Abs Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(x+1)^{2n+1}}{2n+1}}{\frac{(x+1)^{2n-1}}{2n-1}} \right| = \lim_{n\to\infty} \left| \frac{(x+1)^{2n+1}}{2n+1} \cdot \frac{2n-1}{(x+1)^{2n-1}} \right|$$

$$= \lim_{n\to\infty} |(x+1)^2| \left| \frac{2n-1}{2n+1} \right|$$

$$= (x+1)^2 \cancel{\lim_{n\to\infty} \left| \frac{2n-1}{2n+1} \right|}$$

$$= (x+1)^2$$

Converges for $(x+1)^2 < 1 \rightarrow$ $-1 < x+1 < 1$

$-1 \quad -1 \quad -1$

$-2 < x < 0$

Now check $x=-2$ & $x=0$

$\boxed{x=-2}$ converges

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-2+1)^{2n-1}}{2n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^{2n-1}}{2n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)}{2n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n} \frac{1}{2n-1}$$

AST

- $\frac{1}{2n-1} > 0$
- $\frac{1}{2n-1} \geq \frac{1}{2(n+1)-1}$
- $\lim_{n\to\infty} \frac{1}{2n-1} = 0$

So converges.

$\boxed{x=0}$ converges

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1)^{2n-1}}{2n-1}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1}$$

AST

(Same)

So converges

Int of Conv:

$-2 \leq x \leq 0$

OR

$[-2, 0]$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Abs Ratio Test

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x}{\cancel{n!}\cdot(n+1)}\cdot\frac{\cancel{n!}}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$= 0$$

Converges for $0<1 \to$ all x

Int of Conv

$-\infty < x < \infty$ OR all x

OR

$(-\infty, \infty)$

$$\sum_{n=0}^{\infty} n!\,x^n$$

Abs Ratio Test

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n!}(n+1)\cancel{x^n}\,x}{\cancel{n!}\,\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}|(n+1)x|$$

∞ if $x \neq 0$ $\qquad$ 0 if $x = 0$

Int of Conv.

Converges when $\lim_{n\to\infty}|(n+1)x| < 1 \rightarrow \boxed{x = 0}$

Differentiation and Integration of Power Series:

If a power series $\sum c_n (x-a)^n$ converges for $(a-R, a+R)$ for some $R>0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ with domain } (a-R, a+R).$$

Its derivative is

(term-by-term differentiation)

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

Its integral is

(term-by-term integration)

$$\int f(x)\,dx = \left(\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}\right) + C$$

Find a series that represents

$g(x) = \frac{1}{(1-x)^2}$. (for $-1 < x < 1$)

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} \frac{d}{dx}\left[x^n\right] = \frac{d}{dx}\left[\frac{1}{1-x}\right]$$

$$\boxed{\sum_{n=0}^{\infty} n x^{n-1}} = \frac{+1(+1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$= \cancel{0} + 1x^{(1-1)} + 2x^{(2-1)} + 3x^{(3-1)} + \dots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$