Integral Test cont. (8.3)

Last Time

If a_n is a sequence of **nonnegative** terms, then $s_n = \sum_{i=1}^{n} a_i$ is a **nondecreasing** sequence.

Thus $\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n$ converges if and only if it has an upper bound.

Result

$\sum_{i=1}^{\infty} \frac{1}{i}$ diverges b/c it has no upper bound.

I notice that both of these diverge:

$$\sum_{i=1}^{\infty} \frac{1}{i} \quad \& \quad \int_1^{\infty} \frac{1}{x}\,dx$$

Turns out this is true in general:

Integral Test:

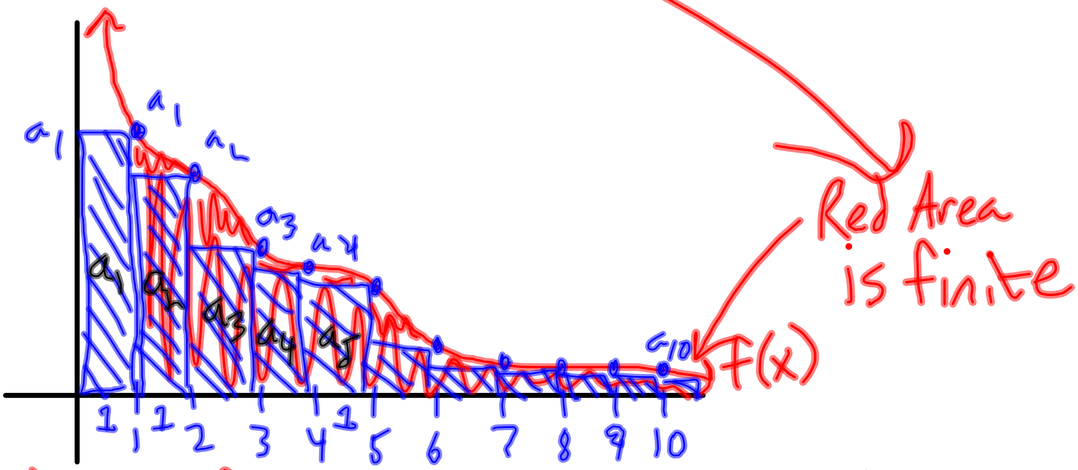
Let a_n be a sequence and f(x) be a continuous, positive, and decreasing function such that $f(i) = a_i$ for all integers i.

1) If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

2) If $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{i=1}^{\infty} a_i$ diverges.

1) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{i=1}^\infty a_i$ converges.

Picture:



The Red Area forms an upper bound for the Blue rectangles (except the first).

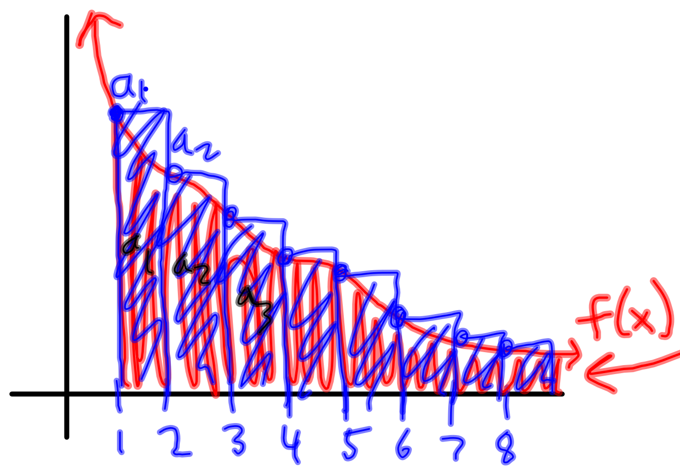
Thus we have an upper bound for the value of $\sum_{i=2}^\infty a_i$, so it converges to a finite value.

finite

And $\sum_{i=1}^\infty a_i = a_1 + \sum_{i=2}^\infty a_i$, so $\sum_{i=1}^\infty a_i$ is finite (converges)

2) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{i=1}^{\infty} a_i$ diverges.

Similarly,



Not finite area

If the blue area had an upper bound for its value, then that upper bound would be an upper bound for the red area. But since the red area diverges, it can't have an upper bound, so the blue area doesn't either.

Thus $\sum_{i=1}^{\infty} a_i \to \infty$

Immediate Results:

$\int_1^\infty \frac{1}{x^p}dx$ diverges for $p \le 1$ $\qquad$ $\int_1^\infty \frac{1}{x^p}dx$ converges for $p > 1$

$\downarrow$ $\qquad$ $\downarrow$

$\sum_{i=1}^\infty \frac{1}{i^p}$ diverges for $p \le 1$ $\qquad$ $\sum_{i=1}^\infty \frac{1}{i^p}$ converges for $p > 1$

Warning:

This does NOT mean

$$\cancel{\int_1^\infty f(x)\,dx = \sum_{i=1}^\infty f(i)}$$

Examples:

Do these series converge or diverge? (Don't care what they converge to.)

$$\sum_{n=1}^{\infty} \frac{1}{12^n}$$

Method 1: Compute series using Geometric Series formula.

$$= \sum_{n=1}^{\infty} \frac{1}{12}\left(\frac{1}{12}\right)^{n-1} = \frac{1/12}{1-1/12} = \frac{1/12}{11/12} = 1/11 \Leftarrow \text{converges}$$

Method 2: Show an upper bound.

$$\sum_{n=1}^{\infty}\left(\frac{1}{12}\right)^n < \sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n = 0.1111\ldots \leftarrow \text{upper bound}$$

Shown an upper bound, so it converges.

$$\sum_{k=1}^{\infty} \frac{5}{k+1}$$

Reindex to

$$\sum_{k=2}^{\infty} \frac{5}{k}$$

This diverges b/c $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges.

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$$ Hint: $\ln(n) \geq 1$ for $n \geq 3$

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}} = \frac{\ln 2}{\sqrt{2}} + \sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}} \geq \frac{\ln 2}{\sqrt{n}} + \underbrace{\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}}_{\text{Diverges b/c } p=\frac{1}{2}\leq 1}$$

So the original diverges.