

Examples of using Convergence Tests on Improper Integrals

Does this improper integral converge or diverge?

$$\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}} dx$$

Idea: Look at the power of x in the denominator.

$$\int_1^{\infty} \frac{1}{x\sqrt{x^2}} dx = \int_1^{\infty} \frac{1}{x^2} dx = \int_1^{\infty} \frac{1}{x^2} dx \leftarrow \text{converges}$$

My guess: it converges.

DCT

$$0 \leq \frac{1}{x\sqrt{x^2-1}} \leq \frac{1}{x\sqrt{x^2}}$$

$$x\sqrt{x^2} \leq x\sqrt{x^2-1}$$

Not true b/c
left denominator is smaller
than right denominator.

LCT

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x\sqrt{x^2-1}}}{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{x^2}{x\sqrt{x^2-1}}$$

Denominator
Too complicated

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{1}{x\sqrt{x^2-1}}} = \lim_{x \rightarrow \infty} \frac{x\sqrt{x^2-1}}{x^2} = \lim_{x \rightarrow \infty} \sqrt{\frac{x^2-1}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \sqrt{\frac{x^2}{x^2}} = 1$$

Is + & finite

So by LCT

$$\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}} dx + \int_1^{\infty} \frac{1}{x^2} dx$$

both converge or both diverge.

$$\int_0^{\infty} \frac{16 \arctan x}{1+x^2} dx$$

Thought: $\arctan x$ is bounded between 0 and $\pi/2$

$$0 \leq \frac{16 \arctan x}{1+x^2} \leq \frac{16(\pi/2)}{1+x^2} = \frac{8\pi}{1+x^2} \leq \frac{8\pi}{x^2}$$

By DCT, if $\int_0^{\infty} \frac{8\pi}{x^2+1} dx$ converges, so does $\int_0^{\infty} \frac{16 \arctan x}{1+x^2} dx$.

Use DCT

- $0 \leq \frac{16 \arctan x}{1+x^2} \leq \frac{16(\pi/2)}{1+x^2} = \frac{8\pi}{1+x^2} \leq \frac{8\pi}{x^2}$
- $\int_0^{\infty} \frac{8\pi}{1+x^2} dx$ converges

Thus

- $\int_0^{\infty} \frac{16 \arctan x}{1+x^2} dx$ converges

$$\int_0^{\infty} 2e^{-\theta} \sin^2(\theta) d\theta \quad 0 \leq \sin^2 \theta \leq 1$$

Hint: ~~$\sin(\theta) \leq \theta$~~ and $e^\theta \geq \theta^3$ for positive theta

$$= \int_0^{\infty} \frac{2 \sin^2 \theta}{e^\theta} d\theta$$

$$e^\theta \geq \theta^3$$

$$1 \geq \frac{\theta^3}{e^\theta}$$

$$\frac{1}{\theta^3} \geq \frac{1}{e^\theta}$$

$$\frac{2 \sin^2 \theta}{e^\theta} \leq \frac{2 \sin^2 \theta}{e^\theta} \leq \frac{2(1)}{e^\theta} \leq \frac{2}{\theta^3}$$

By DCT this converges. Converges

Use DCT

$$0 \leq \frac{2 \sin^2 \theta}{e^\theta} \leq \frac{2(1)}{e^\theta} \leq \frac{2}{\theta^3}$$

$\int_1^{\infty} \frac{2}{\theta^3} d\theta$ converges by power rule

Thus by DCT

$$\int_1^{\infty} \frac{2 \sin^2 \theta}{e^\theta} d\theta \text{ converges.}$$

$$\int_2^{\infty} \frac{1}{\ln x} dx$$

Guess: Diverges

LCT: Compare to $\int_2^{\infty} \frac{1}{x} dx$

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{\ln x}} = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \frac{\infty}{\infty} \leftarrow \text{Indeterminate Form}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1} = \text{Not } +$$

Hint: $\ln x \leq x$ for $x \geq 2$

Use DCT pt 2

$$\bullet 0 \leq \frac{1}{x} \leq \frac{1}{\ln x}$$

$$\bullet \int_2^{\infty} \frac{1}{x} \text{ diverges.}$$

Thus by DCT

$$\bullet \int_2^{\infty} \frac{1}{\ln x} \text{ diverges}$$

