15. Find the Maclaurin Series generated by $f(x) = \dfrac{e^x + e^{-x}}{2}$.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Not method I want:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$$

$$\tfrac{1}{2}(e^x + e^{-x}) = \tfrac{1}{2}\sum_{k=0}^{\infty} \frac{x^k + (-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{1+(-1)^k}{2(k!)} x^k$$

$$f^{(0)}(x) = \frac{e^x + e^{-x}}{2}$$

$$f^{(1)}(x) = \frac{e^x - e^{-x}}{2}$$

$$f^{(2)}(x) = \frac{e^x + e^{-x}}{2}$$

$$\vdots$$

$$f^{(k)}(x) = \frac{e^x + (-1)^k e^{-x}}{2}$$

$$f^{(k)}(0) = \frac{1+(-1)^k}{2}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{1+(-1)^k}{2(k!)} x^k} = 1 + 0 + \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \cdots$$

$$f^{(2k)}(x) = \frac{e^x + e^{-x}}{2} \qquad f^{(2k+1)}(x) = \frac{e^x - e^{-x}}{2}$$

$$f^{(2k)}(0) = \frac{1+1}{2} = 1 \qquad f^{(2k+1)}(x) = \frac{1-1}{2} = 0$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{f^{(2k)}(0)}}{(2k)!} x^{2k} + \cancel{\sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!}} x^{2k+1}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

16. Express $\int e^{x^3}\,dx$ as a power series.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{x^3} = \sum_{k=0}^{\infty} \frac{(x^3)^k}{k!}$$

$$e^{x^3} = \sum_{k=0}^{\infty} \frac{x^{3k}}{k!}$$

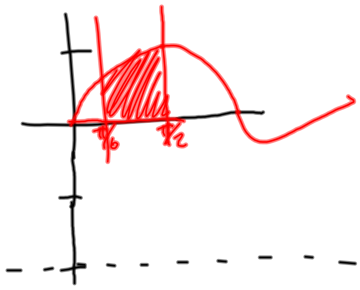
$$\int e^{x^3} dx = \sum_{k=0}^{\infty} \int \frac{x^{3k}}{k!} dx + C$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{3k+1} x^{3k+1} + C$$

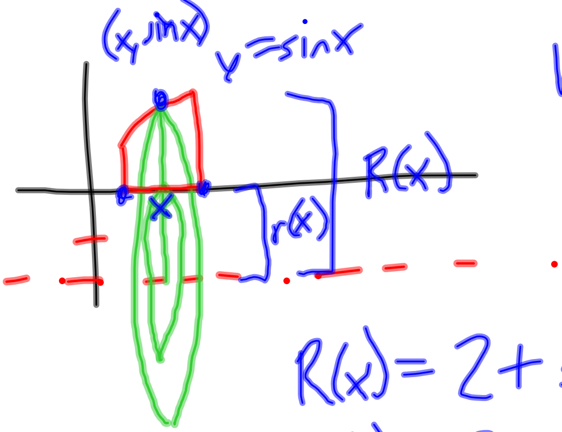
$$= \boxed{\sum_{k=0}^{\infty} \frac{x^{3k+1}}{k!\,(3k+1)}} + C$$

Like #1 & #2

Find the volume of the solid generated by rotating the region bounded by $y=0$, $x=\pi/6$, $x=\pi/2$, and $y=\sin x$, around the line $y=-2$.



① Washer Method



$$V=\int_a^b \pi\left(R(x)^2-r(x)^2\right)dx$$

$$R(x)=2+\sin x$$
$$r(x)=2$$

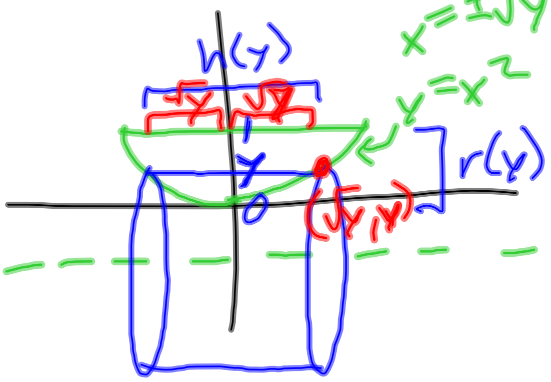
$$V=\int_{\pi/6}^{\pi/2}\pi\left((2+\sin x)^2-4\right)dx$$

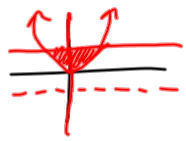
$$=\int_{\pi/6}^{\pi/2}\pi\left(\cancel{4}+4\sin x+\sin^2 x-\cancel{4}\right)dx$$

$$=\int_{\pi/6}^{\pi/2}\pi\left(4\sin x+\sin^2 x\right)dx$$

Rotate Area bounded by $y=1$, and $y=x^2$ around $y=-1$.

Shell Method (#2)





$$V=\int_a^b 2\pi r(y)\,h(y)\,dy$$

$r(y)=1+y$

$h(y)=2\sqrt{y}$

$$V=\int_0^1 2\pi(1+y)2\sqrt{y}\,dy$$

$$=4\pi\int_0^1 y^{1/2}+y^{3/2}\,dy$$

$$=4\pi\left[\tfrac{2}{3}y^{3/2}+\tfrac{2}{5}y^{5/2}\right]_0^1$$

$$=4\pi\left[\tfrac{2}{3}+\tfrac{2}{5}\right]$$

$$=4\pi\left[\tfrac{10}{15}+\tfrac{6}{15}\right]$$

$$=\boxed{\tfrac{64}{15}\pi}$$

14. Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{(4x-1)^n}{\sqrt{n^3}}$.

Abs Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(4x-1)^{n+1}}{(n+1)^{3/2}}}{\frac{(4x-1)^n}{n^{3/2}}} \right| = \lim_{n\to\infty} \left| \frac{(4x-1)^{\cancel{n+1}}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{\cancel{(4x-1)^n}} \right|$$

$$= \lim_{n\to\infty} |4x-1| \left| \left(\frac{n}{n+1}\right)^{3/2} \right|$$

$$= |4x-1|$$

Abs Conv. for $|4x-1|<1$

$$-1<4x-1<1$$

$$0<4x<2$$

$$\boxed{0 \le x \le 1/2}$$

rad of conv. $\boxed{1/4}$

$x=0$ ✓

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

conv.

$x=1/2$ ✓

$$\sum_{n=1}^{\infty} \frac{(2-1)^n}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

conv.

Find $\displaystyle\int \frac{1}{9x^2-25}\,dx$ (#6)

$1-\sin^2\theta=\cos^2\theta$

$1+\tan^2\theta=\sec^2\theta$

$\boxed{\sec^2\theta-1=\tan^2\theta}$

$25\sec^2\theta-25=25\tan^2\theta$

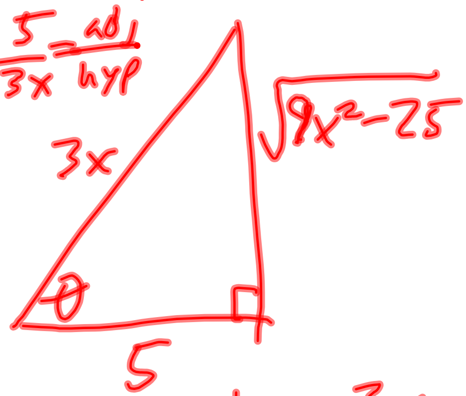
Let $9x^2=25\sec^2\theta$

$3x=5\sec\theta$

$3\,dx=5\sec\theta\tan\theta\,d\theta$

$dx=\frac{5}{3}\sec\theta\tan\theta\,d\theta$

$\cos\theta=\frac{5}{3x}=\frac{adj}{hyp}$



$$=\int \frac{\frac{5}{3}\sec\theta\tan\theta}{25\sec^2\theta-25}\,d\theta$$

$$=\int \frac{\frac{5}{3}\sec\theta\cancel{\tan\theta}}{\cancel{25}_{5}\tan^{\cancel{2}}\theta}\,d\theta$$

$$=\int \frac{1}{15}\,\frac{\sec\theta}{\tan\theta}\,d\theta$$

$$=\int \frac{1}{15}\,\frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\,d\theta=\int \frac{1}{15}\,\frac{1}{\cancel{\cos\theta}}\,\frac{\cancel{\cos\theta}}{\sin\theta}\,d\theta$$

$$=\int \frac{1}{15}\csc\theta\,d\theta$$

$\int\csc\theta=-\ln|\csc\theta+\cot\theta|$

$$=-\frac{1}{15}\ln|\csc\theta+\cot\theta|+C$$

$\csc\theta=\frac{hyp}{opp}=\frac{3x}{\sqrt{9x^2-25}}$

$\cot\theta=\frac{adj}{opp}=\frac{5}{\sqrt{9x^2-25}}$

$$\boxed{=-\frac{1}{15}\ln\left|\frac{3x}{\sqrt{9x^2-25}}+\frac{5}{\sqrt{9x^2-25}}\right|+C}$$

⋮

$$\int\frac{1}{15}\,d\theta=\frac{1}{15}\theta+C$$

$\cos\theta=\frac{5}{3x}$

$\theta=\text{Arccos}\left[\frac{5}{3x}\right]$

$$=\boxed{\frac{1}{15}\text{Arccos}\left[\frac{5}{3x}\right]+C}$$

19. Find the area of the region bounded by the polar equation $r = 4 - 4\cos(\theta)$.

$$A = \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$= \int_0^{2\pi} \tfrac{1}{2}(4-4\cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(16-32\cos\theta+16\cos^2\theta) \, d\theta$$

$$= \int_0^{2\pi} 8-16\cos\theta+8\cos^2\theta \, d\theta$$

$$= \int_0^{2\pi} 8-16\cos\theta+4+4\cos 2\theta \, d\theta$$

$$= \int_0^{2\pi} 12-16\cos\theta+4\cos 2\theta \, d\theta$$

$$= \Big[12\theta-16\sin\theta+2\sin 2\theta\Big]_0^{2\pi}$$

$$= (24\pi-\cancel{16\sin 2\pi}+\cancel{2\sin 4\pi}) - \cancel{(0-0-0)}$$

$$= \boxed{24\pi}$$

20. Find the length of the perimeter of the region bounded by the polar equation $r = 4 - 4\cos(\theta)$.

$\frac{dr}{d\theta} = 4\sin\theta$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{\cancel{16\sin^2\theta} + (16 - 32\cos\theta + \cancel{16\cos^2\theta})}\, d\theta$$

16

$$= \int_0^{2\pi} \sqrt{32 - 32\cos\theta}\, d\theta$$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\sin^2\theta/2 = \frac{1-\cos\theta}{2}$

$64\sin^2\theta/2 = 32 - 32\cos\theta$

$$= \int_0^{2\pi} \sqrt{64\sin^2\theta/2}\, d\theta$$

$$= \int_0^{2\pi} 8\sin\theta/2\, d\theta$$

$$= \left[-16\cos\theta/2\right]_0^{2\pi} = (-16(-1)) - (-16(1))$$

$$= 16 + 16 = \boxed{32}$$

17. Give a polynomial of degree 6 which approximates $f(x) = \sqrt[3]{1+x^3}$. $= (1+x^3)^{1/3}$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

$$(1+x^3)^m = 1 + mx^3 + \frac{m(m-1)}{2!}x^6 + \cdots$$

$$(1+x^3)^{1/3} \approx 1 + \tfrac{1}{3}x^3 + \frac{\tfrac{1}{3}(\cancel{\tfrac{1}{3}-1})^{-2/3}}{2}x^6 \qquad -\frac{\tfrac{1}{3}(\tfrac{2}{3})}{2}$$

$$= \boxed{1 + \tfrac{1}{3}x^3 - \tfrac{1}{9}x^6} \qquad = -\frac{2/9}{2}$$

$$= -\tfrac{2}{9}\cdot\tfrac{1}{2}$$

$$= -\tfrac{1}{9}$$

13. Show that $\sum_{n=0}^{\infty} \frac{(-2)^n}{3^{2n}}$ converges absolutely, converges conditionally, or diverges.

Check Abs Conv.

$$\sum \left|\frac{(-2)^n}{3^{2n}}\right| = \sum \frac{2^n}{3^{2n}} = \sum \frac{2^n}{(3^2)^n} = \sum \frac{2^n}{9^n}$$

$$= \sum_{n=0}^{\infty} (2/9)^n$$

Geo. Series Conv. for $|2/9| < 1$

$\boxed{\text{Abs Conv.}}$

Does $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n^2)}$ conv. abs. conv. cond. OR diverge?

Abs Con.

$$\sum \frac{1}{\ln(n^2)} = \sum \frac{1}{2\ln n} \; ?$$

$\left(\ln n < n \text{ so } \frac{1}{n} < \frac{1}{\ln n} \right)$

DCT

- $0 \le \frac{1}{2n} \le \frac{1}{2\ln n} = \frac{1}{\ln n^2}$
- $\sum \frac{1}{2n}$ diverges

Thus $\sum \frac{1}{\ln n^2}$ diverges.

Check Cond. Conv.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n^2)} = \sum (-1)^n \frac{1}{\ln(n^2)} = \sum (-1)^n \frac{1}{2\ln(n)}$$

AST

- $\frac{1}{2\ln(n)} > 0$ ✓
- $\frac{1}{2\ln(n)} \ge \frac{1}{2\ln(n+1)}$ ✓
- $\frac{1}{2\ln(n)} \to 0$ ✓

Thus $\sum \frac{(-1)}{\ln n^2}$ $\boxed{\text{converges conditionally}}$

12. Show that the series $\sum_{i=1}^{\infty} \frac{2}{3^i} - \frac{2}{3^{i+1}}$ converges or diverges.

$$\boxed{\sum_{n=1}^{\infty} a_n - a_{n+1} = a_1 - \lim_{n\to\infty} a_{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{2}{3^i} - \frac{2}{3^{i+1}} = \frac{2}{3^1} - \cancel{\lim_{n\to\infty} \frac{2}{3^{n+1}}}$$

$$= \boxed{2/3}$$

$\boxed{\text{Converges}}$ to