A <u>unit vector</u> is any vector $\vec{a}$ such that $|\vec{a}| = 1$.

Standard Unit Vectors:

$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

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The <u>unit vector in the direction of a vector $\vec{v}$</u> $\neq \vec{0}$ is given by

$$\frac{\vec{v}}{|\vec{v}|} = \overset{\text{scalar}}{\frac{1}{|\vec{v}|}} \overset{\text{vector}}{\left(\vec{v}\right)}$$

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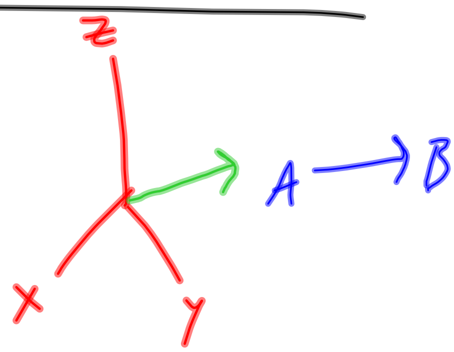
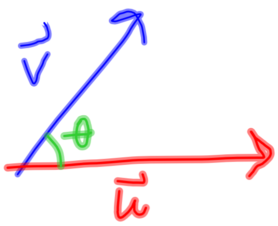
Example: What is the unit vector in the direction of $\overrightarrow{AB}$ where $A = (1,0,1)$ and $B = (3,2,0)$?




$$\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\langle 3-1, 2-0, 0-1 \rangle}{\sqrt{(3-1)^2 + (2-0)^2 + (0-1)^2}}$$

$$= \frac{\langle 2, 2, -1 \rangle}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{\langle 2, 2, -1 \rangle}{\cancel{\sqrt{9}}\ 3} = \langle 2/3, 2/3, -1/3 \rangle$$

Idea: Want the angle between two vectors.




To do this we define the dot product of two vectors.

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

(If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & $\vec{v} = \langle v_1, v_2, v_3 \rangle$)

An alternate definition (using the Law of Cosines)

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta}$$

(where $\theta$ is the angle between the vectors.)

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So the angle between $\vec{u}$ & $\vec{v}$ is:

$$\theta = \operatorname{Arccos}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right) = \operatorname{Arccos}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\vec{u}||\vec{v}|}\right)$$

Ex: Find the Angle between
$\vec{u} = \vec{i} - 2\vec{j} - 2\vec{k}$ and
$\vec{v} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

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$$\theta = \text{Arccos}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$\vec{u} = \langle 1, -2, -2\rangle$
$\vec{v} = \langle 6, 3, 2\rangle$

$$= \text{Arccos}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}||\vec{v}|}\right)$$

$$= \text{Arccos}\left(\frac{\cancel{6} - \cancel{6} - 4}{(3)(7)}\right)$$

$|\vec{u}| = \sqrt{1+4+4}$
$= \sqrt{9} = 3$
$|\vec{v}| = \sqrt{36+9+4}$
$= \sqrt{49} = 7$

$$= \text{Arccos}\left(-\frac{4}{21}\right)$$

$$\approx 1.76 \text{ radians} = 1.76 \frac{180^\circ}{\pi}$$

Two nonzero vectors $\vec{u}$ & $\vec{v}$ are said to be <u>orthoganal / perpendicular</u> if the angle between them is $\pi/2 = 90^\circ$.

This is only true if

$$\boxed{\vec{u} \cdot \vec{v} = 0}$$

Why?

If

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta = 0$$

$$\downarrow$$

$$\cos\theta = 0$$

$$\downarrow$$

$$\theta = \pi/2$$

(We also say that $\vec{0}$ is orthoganal to every vector since $\vec{0} \cdot \vec{u} = 0u_1 + 0u_2 + 0u_3 = 0$)

# Properties of the Dot Product

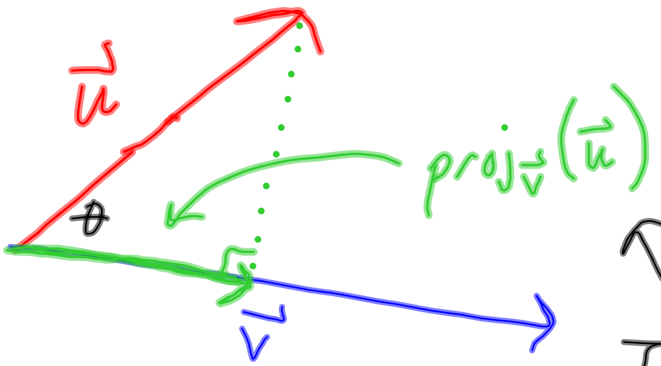
1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $(c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$
3. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
4. $\boxed{\vec{u} \cdot \vec{u} = |\vec{u}|^2}$
5. $\vec{0} \cdot \vec{u} = 0$

Proof of 4

$$\begin{aligned}
\vec{u} \cdot \vec{u} &= u_1 u_1 + u_2 u_2 + u_3 u_3 \\
&= (u_1)^2 + (u_2)^2 + (u_3)^2 \\
&= \left(\sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2}\right)^2 \\
&= \left(|\vec{u}|\right)^2 = |\vec{u}|^2
\end{aligned}$$

$$\begin{aligned}
\vec{u} \cdot \vec{u} &= |\vec{u}||\vec{u}| \cos(\theta_{\vec{u}\vec{u}}) \\
&= |\vec{u}|^2 \cancel{\cos(0)} \\
&= |\vec{u}|^2
\end{aligned}$$

angle between $\vec{u}$ & $\vec{u}$




The projection of the vector $\vec u$ onto $\vec v$.

$$\text{proj}_{\vec v}(\vec u) = \frac{\vec v}{|\vec v|}\,|\vec u|\cos\theta$$

Unit vector in the direction of $\vec v$

length of $\vec u$ in the $\vec v$ direction

$$= \vec v\,\frac{|\vec u||\vec v|\cos\theta}{|\vec v|^2}$$

$$= \boxed{\vec v\left(\frac{\vec u\cdot\vec v}{|\vec v|^2}\right)}$$