Recall: the Taylor Polynomial of order n generated by f(x)
at x=a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Claim: $P_n(x)$ approximate $f(x)$

Taylor's Formula:

If $f(x)$ is infinitely differentiable on an interval I containing the numbers a and x, then there is another number c_n between a & x such that

$$f(x) = P_n(x) + R_n(x)$$

"Remainder" or "Error Term"

where $R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}$

$$f(x) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right) + \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}$$

Notice!

$$f(x) = P_n(x) + R_n(x)$$

$$\downarrow$$

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} P_n(x) + \lim_{n\to\infty} R_n(x)$$

$$f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \lim_{n\to\infty} R_n(x)$$

$$f(x) = \boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k} + \lim_{n\to\infty} R_n(x)$$

Taylor Series

some number between x & a

So if $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1} = 0$

then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Show that $f(x)=e^x$ converges to its Maclaurin series (Taylor series for $a=0$) for all real numbers x.

Taylor's formula shows:

Number between x & a

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{n+1}$$

Note
$$f(x)=e^x \qquad f^{(k)}(x)=e^x$$
$$f'(x)=e^x$$
$$f''(x)=e^x$$
$$\vdots$$

So,
$$f(x)=\sum_{k=0}^{n}\frac{e^{\cancel{a}\,0}}{k!}(x-\cancel{a})^k+\frac{e^{c_n}}{(n+1)!}(x-\cancel{a})^{n+1}$$

$$f(x)=\sum_{k=0}^{n}\frac{x^k}{k!}+e^{c_n}\frac{x^{n+1}}{(n+1)!}$$

$$f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}+\lim_{n\to\infty}e^{c_n}\frac{x^{n+1}}{(n+1)!}$$

(Again, all I need to do is show $R_n(x)\to 0$.)

$$\lim_{n\to\infty}R_n(x)=\lim_{n\to\infty}e^{c_n}\frac{x^{n+1}}{(n+1)!}$$

between x & 0

$$\lim_{n\to\infty}\left|e^{c_n}\frac{x^{n+1}}{(n+1)!}\right|$$

If $x\le 0$
$$\le\lim_{n\to\infty}\left|e^{0}\frac{x^{n+1}}{(n+1)!}\right|$$
$$=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\right|$$

If $x>0$
$$\le\lim_{n\to\infty}\left|e^{x}\frac{x^{n+1}}{(n+1)!}\right|$$
$$=e^x\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\right|$$

Either way, I want to show
$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\right|=\lim_{n\to\infty}\left|\frac{x^{n}}{n!}\right|=0\ \checkmark$$

So
$$\lim_{n\to\infty}\left|e^{c_n}\frac{x^{n+1}}{(n+1)!}\right|\overset{=}{\le}0\ \checkmark$$