Limit Comparison Test for Series

Let $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ be series w/ nonnegative terms.

- If $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ both converge or both diverge.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ Converges.

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$$

diverge

$$\left(\text{Compare to } \sum_{n=1}^{\infty} \frac{2n}{n^2} = \sum_{n=1}^{\infty} \frac{2}{n}\right)$$

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- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

$$\lim_{n\to\infty} \frac{\frac{2n+1}{n^2+2n+1}}{\frac{2}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{2n+1}{n^2+2n+1}\right)\left(\frac{n}{2}\right) = \lim_{n\to\infty} \frac{2n^2+n}{2n^2+4n+2}$$

$$= \lim_{n\to\infty} \frac{2n^2}{2n^2} = 1$$

Since $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, LCT says

$\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$ + $\sum_{n=1}^{\infty} \frac{2}{n}$ ~~both converge~~ or

both diverge.

$$\sum_{n=2}^{\infty} \frac{1+n\cdot\ln(n)}{n^2+5}$$

diverge

$$\left(\text{Compare to } \sum_{n=1}^{\infty} \frac{1}{n}.\right)$$

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$$\lim_{n\to\infty} \frac{\frac{1+n\ln(n)}{n^2+5}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n+n^2\ln(n)}{n^2+5} = \lim_{n\to\infty} \frac{\cancel{n^2}\ln(n)}{\cancel{n^2}}$$

$$= \infty$$

As $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ & $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

we know $\sum_{n=2}^{\infty} \frac{1+n\ln(n)}{n^2+2}$ diverges.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$$

Limit Comparison Test for Series

Let $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ be series w/ nonnegative terms.

- If $0 < \lim_{n \to \infty} \frac{a_n}{b_n} < \infty$, then $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ both converge or both diverge.
- If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
- If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Guess: $\frac{1}{n}$

$$\lim_{n \to \infty} \frac{\frac{\ln n}{n^{3/2}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^1 \cdot \ln n}{n^{3/2}}$$

$$= \lim_{n \to \infty} \frac{\ln(n)}{n^{1/2}} \overset{LH}{=} \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{2} n^{-1/2}} = \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n \to \infty} \frac{1}{\cancel{n}} \cdot 2\cancel{\sqrt{n}}$$

$$\sqrt{n}\sqrt{n}$$

$$\cancel{= \lim_{n \to \infty} \frac{2}{\sqrt{n}} = 0} \leftarrow$$ Didn't help!

Guess: $\sum \frac{1}{n^2}$ ← converges

$$\lim_{n \to \infty} \frac{\frac{\ln n}{n^{3/2}}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2 \ln n}{n^{3/2}} = \cancel{\lim_{n \to \infty} \sqrt{n} \ln n = \infty}$$

Correct Guess: $\sum \frac{1}{n^{3/4}}$ ← converge

$$\lim_{n \to \infty} \frac{\frac{\ln n}{n^{3/2}}}{\frac{1}{n^{3/4}}} = \lim_{n \to \infty} \frac{n^{3/4} \ln n}{n^{6/4}}$$

$$= \lim_{n \to \infty} \frac{\ln n}{n^{1/4}} \overset{LH}{=} \lim_{n \to \infty} \frac{1/n}{1/4\, n^{-3/4}}$$

$$= \lim_{n \to \infty} \frac{1/n}{\frac{1}{4n^{3/4}}} = \lim_{n \to \infty} \frac{4n^{3/4}}{\cancel{n}_{\,n^{1/4}}} = 0$$

As $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ & $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ converges,

I know by LCT $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converges too.