Telescoping Series Test

$$\sum_{i=1}^{\infty}(a_i - a_{i+1}) = \lim_{n\to\infty} \sum_{i=1}^{n}(a_i - a_{i+1})$$

$$= \lim_{n\to\infty} (a_1 - \cancel{a_2}) + (\cancel{a_2} - \cancel{a_3}) + \cancel{\cdot} \cdots \cancel{\cdot} + (\cancel{a_n} - a_{n+1})$$

$$= \lim_{n\to\infty} a_1 - a_{n+1}$$

$$\lim_{n\to\infty} \cancel{(1 - \tfrac{1}{2})} + \cancel{(\tfrac{1}{2} - \tfrac{1}{3})} + \cancel{\cdots +} (\tfrac{1}{n} - \tfrac{1}{n+1})$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{1} - \cancel{\frac{1}{n+1}}$$

$$= 1$$

OR

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\cdots}$$

$$= 1$$

$$= \sum_{n=1}^{\infty} \ln(n+1) - \ln(n) = \infty$$

$$= -\sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$$

$$= -\lim_{n\to\infty} \ln(1) - \ln(n+1)$$

$$= \lim_{n\to\infty} \ln(n+1) = \infty$$

So $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges (to ∞).

By a result on Weds,

$\sum_{n=1}^{\infty} (-1)^n \ln\left(\frac{n+1}{n}\right)$ converges. +

Thus converges conditionally.

Does

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+\sqrt[3]{n}}$$

converge absolutely

converge conditionally

or diverge?

First check $\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{1+n^{1/3}}$

~~Try Root Test!~~

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{1+n^{1/3}}} = \lim_{n\to\infty} \left(\frac{1}{1+n^{1/3}}\right)^{1/n} = \lim_{n\to\infty} \frac{(1)^{1/n}}{(1+n^{1/3})^{1/n}}$$

$$= \lim_{n\to\infty} \frac{1}{(1+n^{1/3})^{1/n}}$$

~~Try something else~~

Direct Comparison

- $0 \le \frac{1}{n} \le \frac{1}{1+n^{1/3}}$ $\left(\text{Ex: } 0 \le \frac{1}{8} \le \frac{1}{1+\sqrt[3]{8}} = \frac{1}{1+2}\right)$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (by p-series, or Harmonic Series)

Thus by DCT, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt[3]{n}}$ diverges.

Now check $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+\sqrt[3]{n}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{1+\sqrt[3]{n}}$

Use AST:

1. $\frac{1}{1+\sqrt[3]{n}}$ is positive ✓
2. $\frac{1}{1+\sqrt[3]{n}} \ge \frac{1}{1+\sqrt[3]{n+1}}$ ✓
3. $\displaystyle\lim_{n\to\infty} \frac{1}{1+\sqrt[3]{n}} = 0$ ✓

So $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+\sqrt[3]{n}}$ converges by AST!

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+\sqrt[3]{n}}$ converges conditionally